ON THE MOTIVE OF THE STACK OF BUNDLES

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ABSTRACT. Let G be a split connected semisimple group over a field. We give a conjectural formula for the motive of the stack of G-bundles over a curve C, in terms of special values of the motivic zeta function of C. The formula is true if $C = \mathbb{P}^1$ or $G = \mathrm{SL}_n$. If $k = \mathbb{C}$, upon applying the Poincaré or Serre characteristic, the formula reduces to results of Teleman and Atiyah-Bott on the gauge group. If $k = \mathbb{F}_q$, upon applying the counting measure, it reduces to the fact that the Tamagawa number of G over the function field of C is $|\pi_1(G)|$.

1. INTRODUCTION

We work over a ground field k. For a variety Y we write $\mu(Y)$ for its class in the K-ring of varieties, $K_0(Var_k)$.

As any principal GL_n-bundle (or GL_n-torsor) $P \to X$ (X a variety) is locally trivial in the Zariski topology, we have the formula $\mu(P) = \mu(X)\mu(\mathrm{GL}_n)$. We will use this fact to define $\mu(\mathfrak{X}) \in \widehat{K}_0(Var_k)$ whenever $\mathfrak{X}$ is an algebraic stack stratified by global quotients. Here $\widehat{K}_0(Var_k)$ is the dimensional completion of $K_0(Var_k)[\frac{1}{\mathbb{L}}]$, in which $\mu(\mathrm{GL}_n)$ is invertible. In fact, if $\mathfrak{X} \cong [X/\mathrm{GL}_n]$ is a global quotient, we define $\mu(\mathfrak{X}) = \frac{\mu(X)}{\mu(\mathrm{GL}_n)}$, and generalize from there. Note that every Deligne-Mumford stack of finite type is stratified by global quotients.

We will also introduce a variation on $\widehat{K}_0(Var_k)$, namely the modified ring $\widehat{K}_0^G(Var_k)$ obtained by imposing the extra relations (the 'torsor relations')

$$\mu(P) = \mu(X)\mu(G),$$

whenever $P \to X$ is a G-torsor and G is a fixed connected split linear algebraic group. One can show that all the usual characteristics factor through this ring. In the appendix, the second author will show that there is a ring homomorphism

$$\widehat{K}_0^G(Var_k) \to K_0(\mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}}(k, \mathbb{Q})),$$

where $\mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}}(k, \mathbb{Q})$ is the $\mathbb{Q}$-linearization of Voevodsky's triangulated category of effective geometrical motives and $\mathrm{char}(k) = 0$.

Throughout this paper C is a smooth projective geometrically connected curve over k. We fix also a split semisimple connected algebraic group G over k. Let $\mathfrak{Bun}_{G,C}$ denote the moduli stack of G-torsors on C. The stack $\mathfrak{Bun}_{G,C}$ is stratified by global quotients, and even though it is not of finite type, its motive still converges in $\widehat{K}_0(Var_k)$, because the dimensions of the boundary strata (where the bundle becomes more and more unstable) tend to $-\infty$.

The purpose of this paper is to propose a conjectural formula for the motive of $\mathfrak{Bun}_{G,C}$ in $\widehat{K}_0^G(Var_k)$. Our formula expresses $\mu(\mathfrak{Bun}_{G,C})$ in terms of special values of the motivic zeta function of C. For simply connected G, the formula reads:

$$\mu(\mathfrak{Bun}_{G,C}) = \mathbb{L}^{(g-1)\dim G} \prod_{i=1}^{r} Z(C, \mathbb{L}^{-d_i}),$$

where the d_i are the numbers one higher than the exponents of G.

If k is a finite field, we can apply the counting measure to this formula. We obtain a statement equivalent to the celebrated conjecture of Weil, to the effect that the Tamagawa number of G (as a group over the function field of C) is equal to 1. Of course, Weil's conjecture is much more general, as it applies to arbitrary semisimple simply connected groups over any global field.

The proof of the Tamagawa number conjecture in the case of a split group induced from the ground field was completed by Harder [Har74] by studying residues of Eisenstein series and using an idea of Langlands. Motivic Eisenstein series have been defined in [Kap00] so it is natural to ask if there is a proof of our conjecture along similar lines.

We consider our conjecture to be a motivic version of Weil's Tamagawa number conjecture. Thus we are lead to consider

$$\tau(G) = \mathbb{L}^{(1-g)\dim G} \mu(\mathfrak{Bun}_{G,C}) \prod_{i=1}^{r} Z(C, \mathbb{L}^{-d_i})^{-1} \quad \in \quad \widehat{K}_0^G(Var_k)$$

as the *motivic Tamagawa number* of G. We hope to find an interpretation of $\tau(G)$ as a measure in a global motivic integration theory, to be developed in the future.

We provide four pieces of evidence for our conjecture:

In Section 4, we prove that if $k = \mathbb{C}$ and we apply the Poincaré characteristic to our conjecture, the simply connected case is true. It follows from results on the Poincaré series of the gauge group of G and the purity of the Hodge structure of $\mathfrak{Bun}_{G,C}$ due to Teleman [Tel98].

In Section 5, we verify that if $k = \mathbb{F}_q$, and we apply the counting measure to our conjecture it reduces to theorems of Harder and Ono

that assert that the Tamagawa number of G is the cardinality of the fundamental group of G.

In Section 6, we prove our conjecture for $G = \mathrm{SL}_n$ using the construction of matrix divisors in [BGL94].

Finally, in Section 7, we prove our conjecture for $C = \mathbb{P}^1$, using the explicit classification of G-torsors due to Grothendieck and Harder.

2. The Motive of an Algebraic Stack

2.1. **Dimensional completion of the K-ring of varieties.** Let k be a field. The underlying abelian group of the ring $K_0(Var_k)$ is generated by the symbols $\mu(X)$, where X is the isomorphism class of a variety over k, subject to all relations

$$\mu(X) = \mu(X \setminus Z) + \mu(Z) \quad \text{if } Z \text{ is closed in } X\,.$$

We call $\mu(X)$ the *motive* of X.

Cartesian product of varieties induces a ring structure on $K_0(Var_k)$. Thus $K_0(Var_k)$ becomes a commutative ring with unit. Let $\mathbb{L}$ denote the class of the affine line in $K_0(Var_k)$.

The ring $\widehat{K}_0(Var_k)$ is obtained by taking the dimensional completion of $K_0(Var_k)$. Explicitly, define $F^m(K_0(Var_k)_{\mathbb{L}})$ to be the abelian subgroup of

$$K_0(Var_k)_{\mathbb{L}} = K_0(Var_k)[\tfrac{1}{\mathbb{L}}]$$

generated by symbols of the form

$$\frac{\mu(X)}{\mathbb{L}^n}$$

where $\dim X - n \leq -m$. This is a ring filtration and $\widehat{K}_0(Var_k)$ is obtained by completing $K_0(Var_k)_{\mathbb{L}}$ with respect to this filtration.

Note that $\mathbb{L}^n - 1$ is invertible in $\widehat{K}_0(Var_k)$ as

$$
\begin{aligned}
\frac{1}{\mathbb{L}^n - 1} &= \frac{1}{\mathbb{L}^n}\left(\frac{1}{1 - \frac{1}{\mathbb{L}^n}}\right) \\
&= \mathbb{L}^{-n}(1 + \mathbb{L}^{-n} + \ldots).
\end{aligned}
$$

Using the Bruhat decomposition one finds that

$$\mu(\mathrm{GL}_n) = (\mathbb{L}^n - 1)(\mathbb{L}^n - \mathbb{L})\ldots(\mathbb{L}^n - \mathbb{L}^{n-1})$$

and hence that the motive of GL_n is invertible in $\widehat{K}_0(Var_k)$. This will be important below. For other groups we are interested in we have:

Proposition 2.1. *Let G be a connected split semisimple group over k. Then*

$$\mu(G) = \mathbb{L}^{\dim G} \prod_{i=1}^{r} (1 - \mathbb{L}^{-d_i})$$

in $\widehat{K}_0(Var_k)$. Here r is the rank of G and the d_i are the numbers one higher than the exponents of G.

Proof. We choose a Borel subgroup B of G with maximal torus T and unipotent radical U. Since T-bundles and U-bundles over varieties are Zariski-locally trivial, we have $\mu(G) = \mu(G/P)\mu(T)\mu(U)$. The torus T is a product of multiplicative groups, so $\mu(T) = (\mathbb{L}-1)^r$. The unipotent group U is an iterated extension of additive groups, so $\mu(U) = \mathbb{L}^u$, where $u = \frac{1}{2}(\dim G - r)$ is the dimension of U. Finally, the flag variety G/B has a cell decomposition coming from the Bruhat decomposition, and we have $\mu(G/B) = \sum_{w \in W} \mathbb{L}^{\ell(w)}$, where W is the Weyl group and $\ell(w)$ the length of a Weyl group element. We have

$$(\mathbb{L} - 1)^r \sum_{w \in W} \mathbb{L}^{\ell(w)} = \prod_{i=1}^{r} (\mathbb{L}^{d_i} - 1),$$

by Page 150 of [Kan00] or Page 155 of [Car72], and hence

$$\mu(G) = \mathbb{L}^u \prod_{i=1}^{r} (\mathbb{L}^{d_i} - 1) = \mathbb{L}^{\dim G} \prod_{i=1}^{r} (1 - \mathbb{L}^{-d_i}),$$

since $u + \sum_{i=1}^{r} d_i = \dim G$, by Solomon's theorem, see Page 320 of [Kan00]. $\qquad\square$

2.2. **The motive of an algebraic stack.** All our algebraic stacks will be Artin stacks, locally of finite type, all of whose geometric stabilizers are linear algebraic groups. We will simply refer to such algebraic stacks as *stacks with linear stabilizers*.

By a result of Kresch (Proposition 3.5.9 in [Kre99]), every stack with linear stabilizers admits a stratification by locally closed substacks all of which are quotients of a variety by GL_n, for various n. Note that unless the stack $\mathfrak{X}$ is of finite type, there is no reason why such a stratification should be finite.

Let us remark that for any stack $\mathfrak{Z}$, the reduced substack $\mathfrak{Z}^{\mathrm{red}} \subset \mathfrak{Z}$ is locally closed, so that $\mathfrak{X}$ and $\mathfrak{X}^{\mathrm{red}}$ have the same stratifications by locally closed reduced substacks.

Definition 2.2. We call a stack $\mathfrak{X}$ with linear stabilizers **essentially of finite type**, if it admits a countable stratification $\mathfrak{X} = \bigcup \mathfrak{Z}_i$, where each $\mathfrak{Z}_i$ is of finite type and $\dim \mathfrak{Z}_i \to -\infty$ as $i \to \infty$.

Every stack with linear stabilizers which is essentially of finite type admits countable stratifications $\mathfrak{X} = \bigcup \mathfrak{Z}_i$, where

$$\lim_{i \to \infty} \dim \mathfrak{Z}_i = -\infty$$

and every $\mathfrak{Z}_i$ is a global quotient of a k-variety X_i by a suitable GL_{n_i}. We call such stratifications *standard*.

Let

$$\mathfrak{X} = \bigcup_{i=0}^{\infty} [X_i/\mathrm{GL}_{n_i}]$$

be a standard stratification of the essentially of finite type stack $\mathfrak{X}$. Define

$$\mu(\mathfrak{X}) = \sum_{i=0}^{\infty} \frac{\mu(X_i)}{\mu(\mathrm{GL}_{n_i})}.$$

Note that the infinite sum converges in $\widehat{K}_0(Var_k)$, by our assumptions.

The next lemma implies that our definition of $\mu(\mathfrak{X})$, the motive of the stack $\mathfrak{X}$, does not depend on the choice of a standard stratification of $\mathfrak{X}$.

Lemma 2.3. *Let $\mathfrak{X} \cong [X/\mathrm{GL}_n]$ be a global quotient stack, where X is a variety. Let $\mathfrak{X} = \bigcup_{i=1}^{N} \mathfrak{Z}_i$ be a stratification of $\mathfrak{X}$ by locally closed substacks $\mathfrak{Z}_i$, which are, in turn, global quotient stacks $\mathfrak{Z}_i \cong [X_i/\mathrm{GL}_{n_i}]$. Then*

$$\frac{\mu(X)}{\mu(\mathrm{GL}_n)} = \sum_{i=1}^{N} \frac{\mu(X_i)}{\mu(\mathrm{GL}_{n_i})}$$

in $\widehat{K}_0(Var_k)$.

Proof. Let Z_i be the preimage of $\mathfrak{Z}_i \subset \mathfrak{X}$ in X under the structure morphism $X \to \mathfrak{X}$. Then $\mathfrak{Z}_i \cong [Z_i/\mathrm{GL}_n]$ and $\mathfrak{Z}_i \cong [X_i/\mathrm{GL}_{n_i}]$. Define Y_i as the fibered product

$$
\begin{array}{ccc}
Y_i & \longrightarrow & Z_i \\
\downarrow & & \downarrow \\
X_i & \longrightarrow & \mathfrak{Z}_i
\end{array}
$$

Then $Y_i \to X_i$ is a principal GL_n-bundle and $Y_i \to Z_i$ is a principal GL_{n_i}-bundle. Since GL_n-bundles are always Zariski-locally trivial, we conclude that $\mu(Y_i) = \mu(X_i)\mu(\mathrm{GL}_n)$ and $\mu(Y_i) = \mu(Z_i)\mu(\mathrm{GL}_{n_i})$. Thus we have

$$\frac{\mu(X)}{\mu(\mathrm{GL}_n)} = \sum_{i=1}^{N} \frac{\mu(Z_i)}{\mu(\mathrm{GL}_n)} = \sum_{i=1}^{N} \frac{\mu(X_i)}{\mu(\mathrm{GL}_{n_i})}$$

as required. $\qquad\square$

According [Kre99, Proposition 3.5.5] the class of algebraic stacks $\mathfrak{X}$ for which $\mu(\mathfrak{X})$ makes sense includes all Deligne-Mumford stacks of finite type.

2.3. **The torsor relations.** An essential ingredient in the definition of the motive of a stack with linear stabilizers was the fact that every GL_n-principal bundle over a variety is Zariski locally trivial. This implies that if $P \to X$ is a principal GL_n-bundle, then

$$(1) \qquad \qquad \mu(P) = \mu(X)\mu(\mathrm{GL}_n) \,,$$

even if X is a stack (where GL_n-bundles are not necessarily Zariski locally trivial any longer).

In Section 7, we will need (1) to hold for more general groups than GL_n. This is why we make the following definition.

Definition 2.4. Fix an algebraic group G. We define $\widehat{K}_0^G(Var_k)$ to be the quotient of the ring $\widehat{K}_0(Var_k)$ by the ideal generated by all elements

$$\mu(P) - \mu(X)\mu(G)$$

where X is a k-variety, and $P \to X$ is a G-torsor.

Lemma 2.5. *Let $\mathfrak{X}$ be an essentially of finite type stack with linear stabilizers and $P \to \mathfrak{X}$ a G-torsor. Then P is also essentially of finite type with linear stabilizers and we have*

$$\mu(P) = \mu(\mathfrak{X})\mu(G)$$

in $\widehat{K}_0^G(Var_k)$.

Example 2.6. Let G be a connected split semisimple group over k. Then we have

$$\mu(BG) = \mathbb{L}^{-\dim G}\prod_{i=1}^{r}(1 - \mathbb{L}^{-d_i})^{-1}$$

in $\widehat{K}_0^G(Var_k)$. Indeed,the torsor relation for G (or Lemma 2.5) implies that we have $\mu(BG) = \mu(G)^{-1}$. Now apply Proposition 2.1.

Remark 2.7. Introducing the torsor relation $\mu(P) = \mu(X)\mu(G)$ for disconnected G kills $\widehat{K}_0(Var_k)$. For example, consider the μ_2-torsor $\mathbb{G}_m \to \mathbb{G}_m$. If char $k \neq 2$, $\mu_2 \cong \mathbb{Z}/2$ and the torsor relation would imply $\mathbb{L} - 1 = 2(\mathbb{L} - 1)$ and hence $1 = 2$, as $\mathbb{L} - 1$ is invertible.

Remark 2.8. For connected G, the ring $\widehat{K}_0^G(Var_k)$ is non-trivial. For example, the ℓ-adic Hodge-Poincaré characteristic (called the Serre characteristic by some authors), factors through $\widehat{K}_0^G(Var_k)$. This follows from the fact that a connected group cannot act non-trivially on

its own ℓ-adic cohomology. By the same token, the singular Hodge-Poincaré characteristic (in case $k = \mathbb{C}$) and the counting measure (in case $k = \mathbb{F}_q$) also factor through $\widehat{K}_0^G(Var_k)$.

Remark 2.9. The second named author of this paper proves in the appendix that the torsor relation (for split and connected linear algebraic groups) holds in Voevodsky's category of effective geometrical motives.

Remark 2.10. Recall that an algebraic group G over k is called *special*, if all its torsors over k-varieties are Zariski-locally trivial. For special groups G, we have $\widehat{K}_0^G(Var_k) = \widehat{K}_0(Var_k)$. Special groups include SL_n and the symplectic groups Sp_{2n}.

One may ask to what extent $\widehat{K}_0^G(Var_k)$ differs from $\widehat{K}_0(Var_k)$, for various groups G.

3. The Main Conjecture

Let G be a split connected semisimple algebraic group over k. We denote by $d_1, d_2, \ldots, d_r$, where r is the rank of G, the numbers one higher than the exponents G. It will be important below that $d_i \geq 2$. Let W be the Weyl group of G and $X(T)$ the character group of a maximal torus T of G. Then W acts on the symmetric algebra of $X(T)$. The d_i are characterized by the fact that the ring of invariants has generators in degrees d_i, see [Che55].

Let C be a smooth projective geometrically connected algebraic curve over k, of genus g. Denote by $C^{(n)}$ the nth symmetric power of C. Recall that the motivic zeta function of C is the power series

$$Z(C, u) = \sum_{n=0}^{\infty} \mu(C^{(n)}) u^n \quad \in \quad \widehat{K}_0(Var_k)[[u]].$$

It is known that this function is in fact rational in u, see [Kap00] and [LL, §3]. The denominator is

$$(1 - u)(1 - \mathbb{L}u)$$

and hence evaluating the zeta function at $u = \mathbb{L}^{-n}$ makes sense when $n \geq 2$.

We denote by $\mathfrak{Bun}_{G,C}$ the moduli stack of G-torsors over C. The motive of $\mathfrak{Bun}_{G,C}$ is defined by the following lemma.

Lemma 3.1. *The stack $\mathfrak{Bun}_{G,C}$ is essentially of finite type with linear stabilizers.*

Proof. See [Beh] or [BD05] for full details. For foundational results on the canonical parabolic the reader is referred to [Beh95]. The automorphism group scheme of a G-bundle E is equal to the scheme of global

sections $\Gamma(C, \operatorname{Aut} G)$, where $\operatorname{Aut}(G)$ is the group scheme over X of automorphisms of E. Since $\operatorname{Aut}(G)$ an affine over C and C is projective, $\Gamma(C, \operatorname{Aut} G)$ is affine, hence linear. Thus $\mathfrak{Bun}_{G,C}$ has linear stabilizers.

Choose a Borel subgroup B of G and call parabolic subgroups of G containing B *standard*. Then every G-torsor E over C has a canonical reduction of structure group F to a uniquely determined standard parabolic $P \subset G$. Thus $E = F \times_P G$. The degree of (the Lie algebra of) the group scheme $\operatorname{Aut}(F) = {}^F P = F \times_{P,ad} P$ is called the *degree of instability* of E. It is a non-negative integer (and 0 if and only if E is semi-stable). Note that we allow G itself to be a parabolic subgroup in this context.

For every $m \geq 0$, the substack $\mathfrak{Bun}^{\leq m} \subset \mathfrak{Bun}_{G,C}$ of torsors of degree of instability less than or equal to m is open in $\mathfrak{Bun}_{G,C}$ and of finite type. The substack $\mathfrak{Bun}^m$ of torsors of degree of instability equal to m is locally closed in $\mathfrak{Bun}_{G,C}$ and of dimension $\dim P(g-1) - m$, which is certainly less than or equal to $\dim G(g-1) - m$, so tends to $-\infty$, as m goes to ∞. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

We now come to our main conjecture.

Conjecture 3.2. *If G is simply connected, we have*

$$\mu(\mathfrak{Bun}_{G,C}) = \mathbb{L}^{(g-1)\dim G} \prod_{i=1}^{r} Z(C, \mathbb{L}^{-d_i})$$

in $\widehat{K}_0^G(Var_k)$.

Remark 3.3. The conjecture makes sense inside the ring $\widehat{K}_0(Var_k)$, but we dare not conjecture its truth in the absence of the torsor relations for G. The proof in the case of $C = \mathbb{P}^1$ uses the torsor relations in an essential way, as we use the formula $\mu(BP)\mu(G) = \mu(G/P)$, for all parabolic subgroups P of G. But note that this requires the torsor relations only for the group G and no others.

Note also, how the formula in Example 2.6 can be thought of as an analogue of our conjecture for C replaced by $\operatorname{Spec} k$. Example 2.6 also relies on the torsor relation for G.

We can generalize the conjecture to arbitrary split connected semisimple G:

Conjecture 3.4. *We have*

$$\mu(\mathfrak{Bun}_{G,C}) = |\pi_1(G)|\, \mathbb{L}^{(g-1)\dim G} \prod_{i=1}^{r} Z(C, \mathbb{L}^{-d_i})$$

in $\widehat{K}_0^G(Var_k)$.

Heuristically, the general case follows from the simply connected case because we expect $\mathfrak{Bun}_{G,C}$ to have $|\pi_1(G)|$ connected components, all with motive equal to the motive of $\mathfrak{Bun}_{\widetilde{G},C}$, where $\widetilde{G}$ is the universal covering group of G.

The rest of this paper is devoted to providing evidence for our conjecture.

4. Evidence from Gauge Field Theory

In this section, $k = \mathbb{C}$. Denote by

$$\chi_c : \widehat{K}_0(Var_{\mathbb{C}}) \to \mathbb{Z}((t^{-1}))$$

the Poincaré characteristic. We will check that Conjecture 3.2 holds after applying χ_c to both sides.

For a smooth $\mathbb{C}$-variety X of dimension n, we have

$$\chi_c(\mu(X)) = \sum_{i,j}(-1)^j \dim W^i H_c^j(X, \mathbb{C})t^i$$

$$= t^{2n}\sum_{i,j}(-1)^j \dim W^i H^j(X, \mathbb{C})t^{-i}$$

$$= t^{2n}P_w(X, t^{-1}),$$

by Poincaré duality, where $P_w(X, t)$ is the Poincaré polynomial of X using *weights*.

The cohomology of a finite type $\mathbb{C}$-stack is endowed with a mixed Hodge structure. It is constructed via simplicial resolutions of the stack. Because every $\mathbb{C}$-stack $\mathfrak{X}$, which is essentially of finite type with linear stabilizers, can be exhausted by finite type open substacks, the cohomology $H^n(\mathfrak{X}, \mathbb{C})$ of $\mathfrak{X}$ also carries a mixed Hodge structure (for every n, the space $H^n(\mathfrak{X}, \mathbb{C})$ is equal to the n-th cohomology of a sufficiently large finite type open substack of $\mathfrak{X}$). Thus, $\mathfrak{X}$ has a Poincaré series

$$P_w(\mathfrak{X}, t) = \sum_{i,j}(-1)^j \dim W^i H^j(\mathfrak{X}, \mathbb{C})t^i.$$

Lemma 4.1. *For every essentially of finite type $\mathbb{C}$-stack with linear stabilizers $\mathfrak{X}$ which is smooth, we have*

$$\chi_c(\mu(\mathfrak{X})) = t^{2\dim\mathfrak{X}}P_w(\mathfrak{X}, t^{-1}).$$

Proof. If $\mathfrak{X} = [X/\mathrm{GL}_n]$ is a global quotient, the formula holds by the Leray spectral sequence for the projection $X \to \mathfrak{X}$.

Suppose $\mathfrak{X}$ is smooth of finite type and $\mathfrak{Z}$ a smooth closed substack. Then $P_w(\mathfrak{X}, t) = P_w(\mathfrak{X} - \mathfrak{Z}, t) + t^{2\operatorname{codim}(\mathfrak{Z},\mathfrak{X})}P_w(\mathfrak{Z}, t)$. This follows easily

from the scheme case by using a simplicial resolution $X_\bullet$ of $\mathfrak{X}$ and the fact that $P_w(\mathfrak{X}, t) = \sum_j (-1)^j P_w(X_j, t) t^j$.

Putting these two remarks together, we get the lemma in the finite type case. For the general case, we choose a stratification $\mathfrak{X} = \bigcup_{i=0}^\infty \mathfrak{Z}_i$, such that every $\mathfrak{X}_n = \bigcup_{i=0}^n \mathfrak{Z}_i$ is a finite type open substack of $\mathfrak{X}$ and $\lim_{i\to\infty} \dim \mathfrak{Z}_i = -\infty$. Then we have

$$\begin{aligned}
\chi_c(\mu(\mathfrak{X})) &= \chi_c(\lim_{n\to\infty} \mu(\mathfrak{X}_n)) \\
&= \lim_{n\to\infty} \chi_c(\mu(\mathfrak{X}_n)) \\
&= t^{2\dim \mathfrak{X}} \lim_{n\to\infty} P_w(\mathfrak{X}_n, t^{-1}) \\
&= t^{2\dim \mathfrak{X}} P_w(\mathfrak{X}, t^{-1}),
\end{aligned}$$

where the last equality follows from the fact that for fixed p, the cohomology group $H^p(\mathfrak{X}_n, \mathbb{Q})$ stabilizes, as $n \to \infty$. $\square$

From [Mac62] we have that

$$\chi_c(Z(C, u)) = \frac{(1+ut)^{2g}}{(1-u)(1-ut^2)}.$$

So because $\mathfrak{Bun}_{G,C}$ is smooth of dimension $\dim G(g-1)$, our conjecture for the simply connected case becomes

$$(2) \qquad P_w(\mathfrak{Bun}_{G,C}, t) = \prod_{i=1}^r \frac{(1+t^{2d_i-1})^{2g}}{(1-t^{2d_i})(1-t^{2(d_i-1)})}.$$

upon applying χ_c to both sides.

The Hodge structure on the cohomology of $\mathfrak{Bun}_{G,C}$ has been computed by Teleman [Tel98]. In fact, Teleman shows (Proposition (4.4) of [ibid.]) that the Hodge structure on $H^*(\mathfrak{Bun}_{G,C})$ is pure, i.e., that the Poincaré series P_w using weights is equal to the Poincaré series P using Betti numbers. Thus we are reduced to computing Betti numbers of $\mathfrak{Bun}_{G,C}$. Atiyah and Bott [AB82] show that

$$H^*(\mathfrak{Bun}_{G,C}) \cong H^*(G)^{\otimes 2g} \otimes H^*(BG) \otimes H^*(\Omega G).$$

It is well known, see for example [Bor53], that we have the following formulas for Poincaré series:

$$\begin{aligned}
P(G, t) &= \prod_{i=1}^r (1+t^{2d_i-1}) \\
P(BG, t) &= \prod_{i=1}^r \frac{1}{1-t^{2d_i}}.
\end{aligned}$$

For the loop group ΩG we have, see [Bot56] or [GR75],

$$P(\Omega G, t) = \prod_{i=1}^{r} \frac{1}{1 - t^{2(d_i - 1)}} \, .$$

This proves the desired formula (2).

Remark 4.2. With no extra effort we can generalize the results of this section to the Hodge-Poincaré or Serre characteristic. Recall that the Serre characteristic $s(X; u, v)$ of a $\mathbb{C}$-variety X is defined as

$$s(X; u, v) = \sum_{i,p,q} (-1)^i h^{p,q} H^i(X, \mathbb{C}) u^p v^q$$

The Serre characteristic is also well-defined for elements of $\widehat{K}_0(Var_{\mathbb{C}})$ and for essentially of finite type $\mathbb{C}$-stacks. If we apply the Serre characteristic to our conjecture (in the simply connected case) we obtain

$$(3) \qquad s(\mathfrak{Bun}_{G,C}; u, v) = \prod_{i=1}^{r} \frac{(1 + u^{d_i} v^{d_i - 1})^g (1 + u^{d_i - 1} v^{d_i})^g}{(1 - u^{d_i} v^{d_i})(1 - u^{d_i - 1} v^{d_i - 1})}$$

This is exactly what Teleman proves in Proposition (4.4) of [Tel98].

5. Evidence from Automorphic Forms

In this section $k = \mathbb{F}_q$. The counting measure $\# : K_0(Var_{\mathbb{F}_q}) \to \mathbb{Z}$ extends to a ring morphism

$$\# : K_0(Var_{\mathbb{F}_q})[\mathbb{L}^{-1}] \longrightarrow \mathbb{Q} \,,$$

but this extension is not continuous, so there is no natural extension of $\#$ to $\widehat{K}_0(Var_{\mathbb{F}_q})$ with values in $\mathbb{R}$. Still, we can make sense of $\#$ on a certain subring of *convergent* motives.

Choose a an embedding $\overline{\mathbb{Q}}_\ell \hookrightarrow \mathbb{C}$. We have the compactly supported Frobenius characteristic

$$F_c : \widehat{K}_0(Var_{\mathbb{F}_q}) \longrightarrow \mathbb{C}((t^{-1})) \,,$$

which is characterized by

$$F_c(\mu X, t) = \sum_{i,j} (-1)^j \operatorname{tr} F_q | W^i H^j_c(\overline{X}, \mathbb{Q}_\ell) t^i \,,$$

for varieties X over $\mathbb{F}_q$. Here $H^j_c(\overline{X}, \mathbb{Q}_\ell)$ is the ℓ-adic étale cohomology with compact supports of the lift $\overline{X}$ of X to the algebraic closure of $\mathbb{F}_q$. The (geometric) Frobenius acting on ℓ-adic cohomology is denoted by F_q.

Definition 5.1. We call an element $x \in \widehat{K}_0(Var_{\mathbb{F}_q})$ with compactly supported Frobenius characteristic $F_c(x, t) = \sum_n a_n t^{-n}$ **convergent** if the series $\sum_n a_n$ converges absolutely in $\mathbb{C}$. If this is the case, we call the sum $\sum_n a_n$ the **counting measure** of x, notation $\#(x)$.

The convergent elements form a subring $\widehat{K}_0(Var_{\mathbb{F}_q})_{\mathrm{conv}}$ of $\widehat{K}_0(Var_{\mathbb{F}_q})$, and we have a well-defined counting measure

$$\# : \widehat{K}_0(Var_{\mathbb{F}_q})_{\mathrm{conv}} \longrightarrow \mathbb{C} \,,$$

which is a ring morphism. Note that $\#$ is not continuous. For example, the sequence $q^n/\mathbb{L}^n$ converges to zero in $\widehat{K}_0(Var_{\mathbb{F}_q})$, but its counting measure converges to 1.

Lemma 5.2. *Every finite type $\mathbb{F}_q$-stack with linear stabilizers $\mathfrak{X}$ has convergent motive $\mu(\mathfrak{X})$. Moreover, $\#(\mu\mathfrak{X})$ is equal to $\#\mathfrak{X}(\mathbb{F}_q)$, the number of rational points of $\mathfrak{X}$ over $\mathbb{F}_q$, counted in the stacky sense, i.e., we count isomorphism classes of the category $\mathfrak{X}(\mathbb{F}_q)$, weighted by the reciprocal of the number of automorphisms.*

Proof. This lemma reduces to the Lefschetz trace formula for F_q on the compactly supported cohomology of an $\mathbb{F}_q$-variety. The reduction uses the simple fact that $\#[X/\mathrm{GL}_n](\mathbb{F}_q) = \#X(\mathbb{F}_q)/\#\mathrm{GL}_n(\mathbb{F}_q)$.   □

Because of the non-continuity of the counting measure, this lemma does not generalize to all essentially finite type stacks over $\mathbb{F}_q$. But we do have a result for certain smooth stacks:

Lemma 5.3. *Let $\mathfrak{X}$ be a smooth stack with linear stabilizers over $\mathbb{F}_q$. Suppose that $\mathfrak{X}$ has a stratification $\mathfrak{X} = \cup_{i=0}^\infty \mathfrak{Z}_i$ by smooth substacks $\mathfrak{Z}_i$, such that for every n the stack $\mathfrak{X}_n = \cup_{i=0}^n \mathfrak{Z}_i$ is an open substack of finite type and*

$$\sum_{n=0}^{\infty} q^{-\operatorname{codim}(\mathfrak{Z}_n, \mathfrak{X})} \sum_{i,j} \dim W^i H^j(\overline{\mathfrak{Z}}_n, \mathbb{Q}_\ell) q^{-i/2} < \infty \,.$$

Then $\mathfrak{X}$ is essentially of finite type, its motive $\mu\mathfrak{X}$ is convergent and $\#(\mu\mathfrak{X}) = \#\mathfrak{X}(\mathbb{F}_q)$.

Proof. Let us emphasize that we assume that for every i, the substack $\mathfrak{Z}_i$ is non-empty and its codimension inside $\mathfrak{X}$ is constant. Let us denote this codimension by c_i.

Let us also remark that our assumptions imply that $\mathfrak{X}$ is essentially of finite type and that $\lim_{n\to\infty} c_n = \infty$. We may also assume, without loss of generality, that the dimension of $\mathfrak{X}$ is constant.

First, we will prove that the trace of the arithmetic Frobenius on the ℓ-adic cohomology of $\mathfrak{X}$ converges absolutely to $q^{-\dim \mathfrak{X}} \#\mathfrak{X}(\mathbb{F}_q)$.

There is a spectral sequence of finite dimensional $\mathbb{Q}_\ell$-vector spaces

$$E_1^{pq} = H^{p+q-2c_p}(\overline{\mathfrak{Z}}_p, \mathbb{Q}_\ell(-c_p)) \Longrightarrow H^{p+q}(\overline{\mathfrak{X}}, \mathbb{Q}_\ell).$$

Even though this is not a first quadrant spectral sequence, we do have that for every n there are only finitely many (p, q) with $p + q = n$ and $E_1^{pq} \neq 0$, so this spectral sequence does converge.

Our assumption on $\mathfrak{X}$ implies that the arithmetic Frobenius Φ_q acting on E_1 has absolutely convergent trace. Thus we get the same result for this trace, no matter in which order we perform the summation. Thus, using the trace formula for the arithmetic Frobenius on finite type smooth stacks with linear stabilizers (see [Beh93]) we have

$$\#\mathfrak{X}(\mathbb{F}_q) = \sum_{p=0}^{\infty} \#\mathfrak{Z}_p(\mathbb{F}_q)$$

$$= \sum_{p=0}^{\infty} q^{\dim \mathfrak{Z}_p} \operatorname{tr} \Phi_q | H^*(\overline{\mathfrak{Z}}_p, \mathbb{Q}_\ell)$$

$$= q^{\dim \mathfrak{X}} \sum_{p=0}^{\infty} \operatorname{tr} \Phi_q | H^*(\overline{\mathfrak{Z}}_p, \mathbb{Q}_\ell(-c_p))$$

$$= q^{\dim \mathfrak{X}} \operatorname{tr} \Phi_q | H^*(\overline{\mathfrak{X}}, \mathbb{Q}_\ell)$$

In particular, we see that $\#\mathfrak{X}(\mathbb{F}_q)$ is finite.

Next we will examine the motive of $\mathfrak{X}$. Note that for smooth stacks of finite type $\mathfrak{Y}$, we have

$$F_c(\mu\mathfrak{Y}, t) = (qt^2)^{\dim \mathfrak{Y}} \Phi(\mathfrak{Y}, t^{-1})$$

where Φ is the Frobenius characteristic defined using the arithmetic Frobenius acting on cohomology without compact supports:

$$\Phi(\mathfrak{Y}, t) = \sum_{i,j} (-1)^j \operatorname{tr} \Phi_q | W^i H^j(\overline{\mathfrak{Y}}, \mathbb{Q}_\ell) t^i.$$

This is essentially Poincaré duality for smooth varieties. Thus we have

$$F_c(\mu\mathfrak{X}, t) = \lim_{n \to \infty} F_c(\mu\mathfrak{X}_n, t)$$

$$= (qt^2)^{\dim \mathfrak{X}} \lim_{n \to \infty} \Phi(\mathfrak{X}_n, t^{-1})$$

$$= (qt^2)^{\dim \mathfrak{X}} \Phi(\mathfrak{X}, t^{-1}).$$

So to prove that $\mu\mathfrak{X}$ is convergent, we need to prove that

$$\sum_i \left| \sum_j (-1)^j \operatorname{tr} \Phi_q | W^i H^j(\overline{\mathfrak{X}}, \mathbb{Q}_\ell) \right| < \infty.$$

But our spectral sequence implies that

$$\sum \dim W^i H^j(\overline{\mathfrak{X}}, \mathbb{Q}_\ell) q^{-i/2} < \infty\,,$$

which is a stronger statement. So we see that $\mu\mathfrak{X}$ is, indeed, convergent and its counting measure takes the value

$$\#(\mu\mathfrak{X}) = q^{\dim \mathfrak{X}} \operatorname{tr} \Phi_q | H^*(\overline{\mathfrak{X}}, \mathbb{Q}_\ell)\,.$$

This we have seen above to be equal to $\#\mathfrak{X}(\mathbb{F}_q)$.   □

We say that a morphism of stacks $\mathfrak{Z} \to \tilde{\mathfrak{Z}}$ is a universal homeomorphism if it is representable, finite, surjective and radical.

Lemma 5.4. *Lemma 5.3 is still valid if we only assume the morphisms* $\mathfrak{Z}_i \to \mathfrak{X}$ *to be universal homeomorphisms onto their image.*

Proof. Let $Z \to X$ be a morphism of finite type smooth schemes which factors as $Z \to \tilde{Z} \to X$, where $\pi : Z \to \tilde{Z}$ is a universal homeomorphism and $i : \tilde{Z} \to X$ a closed immersion with complement U. We have a long exact sequence

$$\ldots \to H^*(\tilde{Z}, i^!\mathbb{Q}_\ell) \to H^*(X, \mathbb{Q}_\ell) \to H^*(U, \mathbb{Q}_\ell) \to \ldots$$

Let $c = \dim X - \dim Z$. We have

$$H^{*-2c}(Z, \mathbb{Q}_\ell(-c)) = H^*(Z, \pi^! i^! \mathbb{Q}_\ell)$$

because Z and X are smooth. Now pulling back via π induces an isomorphism of étale sites (see [Gro61, Expose IX,4.10]). As π_* is the right adjoint of π^*, it is the inverse of π^* and hence also a left adjoint of π^*. Since π is proper, we conclude that $\pi^! = \pi^*$. Thus, we have

$$H^*(Z, \pi^! i^! \mathbb{Q}_\ell) = H^*(Z, \pi^* i^! \mathbb{Q}_\ell) = H^*(\tilde{Z}, i^! \mathbb{Q}_\ell)\,,$$

Thus we have a natural long exact sequence

$$\ldots \to H^{*-2c}(Z, \mathbb{Q}_\ell(-c)) \to H^*(X, \mathbb{Q}_\ell) \to H^*(U, \mathbb{Q}_\ell) \to \ldots$$

This result extends to stacks and filtrations of schemes and stacks consisting of more than two pieces.   □

Lemma 5.5. *The motive of* $\mathfrak{Bun}_{G,C}$ *is convergent. Moreover,* $\#\mu(\mathfrak{Bun}_{G,C}) = \#\mathfrak{Bun}_{G,C}(\mathbb{F}_q)$.

Proof. The hypotheses of Lemma 5.3, or rather its generalization 5.4, are satisfied by the stack $\mathfrak{Bun}_{G,C}$. We may consider the strata $\mathfrak{Bun}_{G,C}^{P,m}$, which contain the bundles E which canonically reduce to the standard parabolic P of G and whose degree of instability is equal to m, see [Beh95]. These strata are not known to be smooth, but the canonical morphism $\mathfrak{Bun}_{P,C}^{ss,m} \to \mathfrak{Bun}_{G,C}^{P,m}$ is a universal homeomorphism. Here

$\mathfrak{Bun}_{P,C}^{ss,m}$ is the open substack of $\mathfrak{Bun}_{P,C}$ consisting of semi-stable bundles of positive (multi-)degree, giving rise to degree of instability m when extending the group to G.

If H is the quotient of P by its unipotent radical, the induced morphism $\mathfrak{Bun}_{P,C}^{ss,m} \to \mathfrak{Bun}_{H,C}^{ss,m}$ induces an isomorphism on ℓ-adic cohomology, because it is an iterated torsor for vector bundle stacks.

This leaves us with proving the convergence of

$$\sum_P \sum_{m=1}^{\infty} q^{-m+(1-g)\dim R_u P} \sum_{i,j} \dim W^i H^j(\overline{\mathfrak{Bun}}_{H,C}^{ss,m}, \mathbb{Q}_\ell) q^{-i/2} .$$

This is not difficult to do using the fact that for fixed H, all $\mathfrak{Bun}_{H,C}^{ss,m}$ are isomorphic to a finite set among them. $\qquad\square$

By this lemma, both sides of our conjectured formula are in the subring $\widehat{K}_0(Var_{\mathbb{F}_q})_{\mathrm{conv}}$. We can thus apply the counting measure $\#$ to our conjecture. Doing this we obtain:

$$(4) \qquad \#\mathfrak{Bun}_{G,C}(\mathbb{F}_q) = |\pi_1(G)| q^{(g-1)\dim G} \prod_{i=1}^{r} \zeta_K(d_i)$$

Here $\zeta_K(s)$ is the usual zeta function of the function field K of the curve C over $\mathbb{F}_q$. It is obtained from the motivic zeta function $Z(C,u)$ of the curve C by applying the counting measure and making the substitution $u = q^{-s}$.

Formula (4) is classical, at least in the simply connected case. Let us recall how it is proved. We consider the adèle ring $\mathbb{A}_K$ of the global field K and notice that the groupoid $\mathfrak{Bun}_{G,C}(\mathbb{F}_q)$ is equivalent to the transformation groupoid of the action of $G(K)$ on $G(\mathbb{A}_K)/\mathfrak{K}$, where $\mathfrak{K} = \prod_{P \in C} G(\widehat{\mathcal{O}}_{C,P})$ is the canonical maximal compact subgroup of $G(\mathbb{A}_K)$.

The transformation groupoid of the $G(K)$-action on $G(\mathbb{A}_K)/\mathfrak{K}$ is equivalent to the transformation groupoid of the $\mathfrak{K}$-action on $G(K)\backslash G(\mathbb{A}_K)$. The groupoid number of points of the latter transformation groupoid can be calculated as

$$(5) \qquad \frac{\mathrm{vol}(G(K)\backslash G(\mathbb{A}_K))}{\mathrm{vol}\,\mathfrak{K}} ,$$

where vol denotes any Haar measure on the locally compact group $G(\mathbb{A}_K)$. This is a simple measure theoretic argument using σ-additivity.

There is a standard normalization of the Haar measure on $G(\mathbb{A}_K)$ known as the Tamagawa measure. With respect to this measure the numerator of (5) is known as the Tamagawa number of G, notation

$\tau(G)$. We conclude that

$$\#\mathfrak{Bun}_{G,C}(\mathbb{F}_q) = \tau(G)\mathrm{vol}(\mathfrak{K})^{-1}\,.$$

The volume of the maximal compact $\mathfrak{K}$ with respect to the Tamagawa measure is easily calculated. We get

$$\mathrm{vol}(\mathfrak{K}) = q^{(1-g)\dim G} \prod_{i=1}^{r} \zeta_K(d_i)^{-1}\,,$$

see [BD05], and thus

$$\#\mathfrak{Bun}_{G,C}(\mathbb{F}_q) = \tau(G)q^{(g-1)\dim G} \prod_{i=1}^{r} \zeta_K(d_i)\,.$$

Comparing this with our conjecture (4) we see that the conjecture becomes equivalent to

(6) $$\tau(G) = |\pi_1(G)|\,.$$

In the simply connected case, the fact $\tau(G) = 1$ was proved by Harder [Har74]. The results of [Ono65] remain true in the function field case (see [BD05]) and from these it follows that the Tamagawa number of a general connected split semisimple group G is equal to $|\pi_1(G)|$.

6. The Case of Sl_n

In the section we prove our conjecture in the case where the group is $G = \mathrm{SL}_n$. Recall that the exponents of SL_n are $2, 3, \ldots, n$. Thus our conjecture states that

$$\mu(\mathfrak{Bun}_{\mathrm{SL}_n,C}) = \mathbb{L}^{(n^2-1)(g-1)} \prod_{i=2}^{n} Z(C, \mathbb{L}^{-i})\,.$$

To calculate the motive of $\mathfrak{Bun}_{\mathrm{SL}_n}$, note that the inclusion $\mathrm{SL}_n \hookrightarrow \mathrm{GL}_n$ defines a morphism of stacks $\mathfrak{Bun}_{\mathrm{SL}_n,C} \to \mathfrak{Bun}_{\mathrm{GL}_n,C}$, whose image is a smooth closed substack $\mathfrak{Bun}_{\det}$ of $\mathfrak{Bun}_{\mathrm{GL}_n}$. Moreover, $\mathfrak{Bun}_{\mathrm{SL}_n}$ is a $\mathbb{G}_m$-bundle over $\mathfrak{Bun}_{\det}$. Thus we have

$$\mu(\mathfrak{Bun}_{\mathrm{SL}_n}) = (\mathbb{L} - 1)\mu(\mathfrak{Bun}_{\det})\,.$$

We can interpret $\mathfrak{Bun}_{\det}$ is the stack of vector bundles over C with trivial determinant.

We will use the construction of matrix divisors in [BGL94]. Let D be an effective divisor on C. We denote by $\mathrm{Div}(D)$ the Quot scheme parameterizing subsheaves

$$E \hookrightarrow \mathcal{O}_C(D)^n\,,$$

where E is a locally free sheaf of rank n and degree 0 on C. The scheme $\mathrm{Div}(D)$ is smooth and proper of dimension $n^2 \deg D$.

Let $\mathrm{Div}_{\det}(D) \subset \mathrm{Div}(D)$ by the closed subscheme defined by requiring the determinant of E to be trivial. This is a smooth subscheme of codimension g. (See [Dhied] for the proof of this.)

Now let us fix, for the moment, an integer $m \geq 0$ and consider the finite type open substack $\mathfrak{Bun}_{\det}^{\leq m}$, of bundles whose degree of instability is at most m. Let D be an effective divisor of sufficiently high degree, such that $H^1(E, \mathcal{O}(D)^n) = 0$, for all bundles E in $\mathfrak{Bun}_{\det}^{\leq m}$. Then the vector spaces $\mathrm{Hom}(E, \mathcal{O}(D)^n)$, for $E \in \mathfrak{Bun}_{\det}^{\leq m}$, are the fibres of a vector bundle $W^{\leq m}(D)$ over $\mathfrak{Bun}_{\det}^{\leq m}$. The rank of this vector bundle is $n^2(\deg D + 1 - g)$.

Let $W_0^{\leq m}(D) \subset W^{\leq m}(D)$ be the open locus of injective maps $E \to \mathcal{O}(D)^n$. Note that

$$W_0^{\leq m}(D) = \mathrm{Div}_{\det}^{\leq m}(D)$$

is the open subvariety of $\mathrm{Div}_{\det}(D)$ parameterizing subsheaves $E \subset \mathcal{O}(D)^n$ of degree of instability at most m.

Lemma 6.1. *Let E and F be vector bundles of equal rank on C. Let D be an effective divisor on C such that $H^1(E, F(D))$ vanishes. Then the locus of the non-injective maps inside $\mathrm{Hom}(E, F)$ has codimension at least $\deg D$.*

Proof. This is proved in Lemma 8.2 of [BGL94]. □

This lemma implies that

$$\lim_{\deg D \to \infty} \frac{\mu(W_0^{\leq m})}{\mathbb{L}^{n^2(\deg D + 1 - g)}} = \lim_{\deg D \to \infty} \frac{\mu(W^{\leq m})}{\mathbb{L}^{n^2(\deg D + 1 - g)}}$$

inside $\widehat{K}_0(Var_k)$. Thus we have

$$\mu(\mathfrak{Bun}_{\det}) = \lim_{m\to\infty} \mu(\mathfrak{Bun}_{\det}^{\leq m})$$

$$= \lim_{m\to\infty} \lim_{\deg D\to\infty} \frac{\mu(W^{\leq m}(D))}{\mathbb{L}^{n^2(\deg D+1-g)}}$$

$$= \lim_{m\to\infty} \lim_{\deg D\to\infty} \frac{\mu(\mathrm{Div}_{\det}^{\leq m}(D))}{\mathbb{L}^{n^2(\deg D+1-g)}}$$

$$= \lim_{\deg D\to\infty} \lim_{m\to\infty} \frac{\mu(\mathrm{Div}_{\det}^{\leq m}(D))}{\mathbb{L}^{n^2(\deg D+1-g)}}$$

$$= \lim_{\deg D\to\infty} \frac{\mu(\mathrm{Div}_{\det}(D))}{\mathbb{L}^{n^2(\deg D+1-g)}}\ .$$

Therefore, the conjecture translates into

$$\lim_{\deg D\to\infty} \frac{\mu(\mathrm{Div}_{\det}(D))}{\mathbb{L}^{n^2(\deg D+1-g)}} = \frac{\mathbb{L}^{(n^2-1)(g-1)}}{\mathbb{L}-1} \prod_{i=2}^{n} Z(C,\mathbb{L}^{-i})$$

or, in other words,

$$(7) \quad \lim_{\deg D\to\infty} \frac{\mu(\mathrm{Div}_{\det}(D))}{\mathbb{L}^{n^2(\deg D+1-g)}}$$

$$= \frac{\mathbb{L}^{(n^2-1)(g-1)}}{\mathbb{L}-1} \sum_{\mathbf{m}=(m_2,\ldots,m_n)} \mu(C^{(\mathbf{m})})\mathbb{L}^{-\sum_{i=2}^{n} im_i}\ .$$

Here the sum ranges over all $(n-1)$-tuples of non-negative integers and we use the abbreviation $C^{(\mathbf{m})} = C^{(m_2)} \times \ldots \times C^{(m_n)}$.

It remains to calculate the motive of $\mathrm{Div}_{\det}(D)$. This we will do by using the stratification induced by a suitable $\mathbb{G}_m$-action via the results of Białynick-Birula [BB73]. Note that we can neglect strata whose codimension goes to infinity, as $\deg D$ goes to infinity.

Consider the action of the torus $\mathbb{G}_m^n$ on $\mathrm{Div}(D)$ induced by the canonical action on the vector bundle $\mathcal{O}_C(D)^n$. It restricts to an action of $\mathbb{G}_m^n$ on $\mathrm{Div}_{\det}(D)$.

The fixed points of $\mathbb{G}_m^n$ on $\mathrm{Div}(D)$ correspond to inclusions of the form

$$\bigoplus_{i=1}^{n} \mathcal{O}_C(D-E_i) \hookrightarrow \mathcal{O}_C(D)^n\,,$$

where $E_1,\ldots,E_n$ are effective divisors with $\sum \deg E_i = n \deg D$ (see [BGL94]). Thus, the components of the fixed locus in $\mathrm{Div}(D)$ are indexed by ordered partitions $\mathbf{m}' = (m_1,\ldots,m_n)$ of $n \deg D$ and the component indexed by $\mathbf{m}'$ is isomorphic to

$$C^{(\mathbf{m}')} = C^{(m_1)} \times \ldots \times C^{(m_n)}\,.$$

The intersection of the fixed component $C^{(\mathbf{m}')}$ with the subvariety $\mathrm{Div}_{\det}(D)$ is given by the condition that $\sum E_i$ be linearly equivalent to nD. Thus, if $m_1 > 2g-2$, this intersection is a projective space bundle with fibre $\mathbb{P}^{m_1-g}$ over $C^{(\mathbf{m})}$, where $\mathbf{m} = (m_2, \ldots, m_n)$. So the motive of the fixed component of $\mathrm{Div}_{\det}(D)$ indexed by $\mathbf{m}'$ is given by

$$\frac{\mathbb{L}^{m_1-g+1}-1}{\mathbb{L}-1}\mu(C^{(\mathbf{m})}).$$

We will see below, that we can neglect the fixed components indexed by $\mathbf{m}'$ with $m_1 \leq 2g-2$.

Now, consider the $\mathbb{G}_m$-action induced by the one-parameter subgroup $\mathbb{G}_m \to \mathbb{G}_m^n$ given by

$$t \longmapsto (t^{\lambda_1}, \ldots, t^{\lambda_n}),$$

where $(\lambda_1, \ldots, \lambda_n)$ is any strictly increasing sequence of integers $\lambda_1 < \ldots < \lambda_n$. The fixed locus of $\mathbb{G}_m$ on $\mathrm{Div}(D)$ is then the same as that of the whole torus $\mathbb{G}_m^n$. We will study the strata

$$X_{\mathbf{m}'}^+ = \{x \in \mathrm{Div}(D) \mid \lim_{t \to 0} tx \in C^{(\mathbf{m}')}\}.$$

and

$$Y_{\mathbf{m}'}^+ = \{x \in \mathrm{Div}_{\det}(D) \mid \lim_{t \to 0} tx \in C^{(\mathbf{m}')} \cap \mathrm{Div}_{\det}(D)\}.$$

There is a morphism $X_{\mathbf{m}'}^+ \to C^{(\mathbf{m}')}$ making $X_{\mathbf{m}'}^+$ into a Zariski locally trivial affine space bundle over $C^{(\mathbf{m}')}$, see [BB73]. The rank of this fibration is the same as the rank of the subbundle N^+ of N on which $\mathbb{G}_m$ acts with positive weights. Here N is the normal bundle of $C^{(\mathbf{m}')}$ inside $\mathrm{Div}(D)$.

The tangent space inside $\mathrm{Div}(D)$ to the fixed point P given by $(E_1, \ldots, E_n) \in C^{(\mathbf{m}')}$ is equal to $\bigoplus_{i,j} \mathrm{Hom}(\mathcal{O}_C(D-E_i), \mathcal{O}_{E_j})$ and the torus $\mathbb{G}_m^n$ acts on the summand $\mathrm{Hom}(\mathcal{O}_C(D-E_i), \mathcal{O}_{E_j})$ through the character $\chi_i - \chi_j$, where χ_i is given by the i-th projection $\chi_i : \mathbb{G}_m^n \to \mathbb{G}_m$. Thus we see that the fibre of N^+ over P is

$$N_P^+ = \bigoplus_{i>j} \mathrm{Hom}(\mathcal{O}_C(D-E_i), \mathcal{O}_{E_j}),$$

and so the rank of N^+ is equal to $\sum_{i=1}^n (n-i)m_i$.

If P is in the subvariety $\mathrm{Div}_{\det}(D)$, the tangent space to P inside $\mathrm{Div}_{\det}(D)$ is the kernel of the diagonal part of the boundary map

$$\bigoplus_{i,j} \mathrm{Hom}(\mathcal{O}_C(D-E_i), \mathcal{O}_{E_j}) \longrightarrow \bigoplus_{i,j} H^1(C, \mathcal{O}_C(E_i - E_j))$$

coming from the universal exact sequence

$$0 \longrightarrow \bigoplus_{i=1}^n \mathcal{O}_C(D-E_i) \longrightarrow \mathcal{O}_C(D)^n \longrightarrow \bigoplus_{i=1}^n \mathcal{O}_{E_i} \longrightarrow 0.$$

(This is proved in [Dhied].) It follows that the rank of the fibration

$$Y^+_{\mathbf{m}'} \longrightarrow C^{(\mathbf{m}')} \cap \mathrm{Div}_{\det}(D)$$

is equal to $\sum_{i=1}^n (n-i)m_i$ as well.

Now we see that the biggest stratum corresponds to an index $\mathbf{m}'$ where m_1 attains the maximal value $n \deg D$. The dimension of all strata coming from $X_{\mathbf{m}'}$ or $Y_{\mathbf{m}'}$ with $m_1 \leq 2g - 2$ is therefore bounded from above by

$$\dim C^{(\mathbf{m}')} + (n-1)(2g-2) + (n-2)(n \deg D - (2g-2))$$
$$= n(n-1) \deg D + 2g - 2.$$

Hence their codimension inside $\mathrm{Div}_{\det}(D)$ is bounded from below by

$$n^2 \deg D - g - n(n-1) \deg D - (2g-2) = n \deg D - 3g + 2$$

which, indeed, goes to infinity with $\deg D$. We conclude that, up to terms we are going to neglect, we have

$$\mu(\mathrm{Div}_{\det}(D)) \approx \sum_{\mathbf{m}'} \frac{\mathbb{L}^{m_1 - g + 1} - 1}{\mathbb{L} - 1} \mu(C^{(\mathbf{m})}) \mathbb{L}^{\sum_{i=1}^m (n-i)m_i},$$

the sum ranging over all $\mathbf{m}' = (m_1, \ldots, m_n)$ with $\sum_{i=1}^n m_i = n \deg D$. We can rewrite this as

$$\frac{\mu(\mathrm{Div}_{\det}(D))}{\mathbb{L}^{n^2(\deg D + 1 - g)}}$$
$$\approx \sum_{\mathbf{m}} \frac{\mathbb{L}^{-\sum_{i=2}^n m_i} - \mathbb{L}^{-n \deg D + g - 1}}{\mathbb{L} - 1} \mu(C^{(\mathbf{m})}) \mathbb{L}^{(n^2-1)(g-1) + \sum_{i=2}^n (1-i)m_i}$$

where the sum ranges over all $\mathbf{m} = (m_2, \ldots, m_n)$ with $\sum_{i=2}^n m_i \leq n \deg D - 2g + 2$.

As $\deg D$ goes to infinity this becomes an equality, in fact, Equation (7), which we set out to prove.

7. THE CASE OF $\mathbb{P}^1$

In this section we use the Grothendieck-Harder classification of torsors on $\mathbb{P}^1$ to prove the conjecture in the special case that $C = \mathbb{P}^1$.

We fix a split maximal torus T inside G and let W be the Weyl group. Let $\mathrm{X}^*(T)$ (resp. $\mathrm{X}_*(T)$) be the character (resp. cocharacter) lattice. We have the root system $\Phi \subset \mathrm{X}^*(T)$ and its dual $\Phi^\vee \subset \mathrm{X}_*(T)$.

We also choose a Borel subgroup B containing T. It determines bases Δ of Φ and $\Delta^\vee$ of $\Phi^\vee$. Denote by $\mathrm{X}_*(T)_{\mathrm{dom}}$ the dominant cocharacters with respect to B. Recall that $\lambda \in \mathrm{X}_*(T)$ is dominant if and only if $(\lambda, \alpha) \geq 0$, for all $\alpha \in \Delta$. The set $\mathrm{X}_*(T)_{\mathrm{dom}}$ is partially ordered:

$\lambda_1 \leq \lambda_2$ if and only if $\lambda_2 - \lambda_1$ is a positive integral linear combination of elements of $\Delta^\vee$.

For a dominant cocharacter $\lambda \in \mathrm{X}_*(T)_{\mathrm{dom}}$, denote by

$$E_\lambda = \mathcal{O}(1) \times_{\mathbb{G}_m, \lambda} G$$

the G-bundle associated to the $\mathbb{G}_m$-bundle $\mathcal{O}(1)$ via the homomorphism $\lambda : \mathbb{G}_m \to G$. (Think of the line bundle $\mathcal{O}(1)$ as a $\mathbb{G}_m$-bundle.)

Proposition 7.1. *Every G-bundle over $\mathbb{P}^1_K$, for a field K/k, becomes isomorphic to E_λ, for a unique $\lambda \in \mathrm{X}_*(T)_{\mathrm{dom}}$, after lifting it to the algebraic closure of K.*

Proof. This result is obtained by combining the Grothendieck-Harder classification of Zariski-locally trivial G-torsors by $\mathrm{X}_*(T)_{\mathrm{dom}}$ with the theorem of Steinberg, to the effect that on $\mathbb{P}^1$ over an algebraically closed field, all G-torsors are Zariski-locally trivial. See also Theorem 4.2 and Proposition 4.3 of [Ram83]. $\qquad\square$

By Proposition 7.1, the bundles E_λ, for $\lambda \in \mathrm{X}_*(T)_{\mathrm{dom}}$, give a complete set of representatives for the points of the stack $\mathfrak{Bun}_{G,\mathbb{P}^1}$. Hence every point of $\mathfrak{Bun}_{G,\mathbb{P}^1}$ is k-rational and its residual gerbe is trivial, equal to $B \operatorname{Aut} E_\lambda$.

Recall that for $\mathfrak{X}$, a locally of finite type algebraic stack over k with set of points $|\mathfrak{X}|$, there is a topology on $|\mathfrak{X}|$, the Zariski topology, such that open substacks of $\mathfrak{X}$ are in bijection to open subsets of $|\mathfrak{X}|$.

Let us identify $|\mathfrak{Bun}_{G,\mathbb{P}^1}|$ with $\mathrm{X}_*(T)_{\mathrm{dom}}$.

Proposition 7.2 (Ramanathan). *Let $\lambda \in \mathrm{X}_*(T)_{\mathrm{dom}}$. Then the set of all $\mu \in \mathrm{X}_*(T)_{\mathrm{dom}}$ with $\mu \leq \lambda$ is open in the Zariski topology on $|\mathfrak{Bun}_{G,\mathbb{P}^1}|$.*

Proof. This is the content of Theorem 7.4 in [Ram83]. $\qquad\square$

It follows from this that the substack of $\mathfrak{Bun}_{G,\mathbb{P}^1}$ of torsors isomorphic to E_λ is locally closed. Moreover, this substack is necessarily equal to the substack $B \operatorname{Aut} E_\lambda$, because a monomorphism of reduced algebraic stacks which is surjective on points is an isomorphism. Thus we have that

$$\mathfrak{Bun}_{G,\mathbb{P}^1} = \bigcup_{\lambda \in \mathrm{X}_*(T)_{\mathrm{dom}}} B \operatorname{Aut} E_\lambda$$

is a stratification of $\mathfrak{Bun}_{G,\mathbb{P}^1}$.

To calculate the motive of $B \operatorname{Aut} E_\lambda$, fix the dominant cocharacter λ. Denote by P the parabolic subgroup of G defined by λ and by U its unipotent radical. The group P is generated by T and all root groups U_α, $\alpha \in \Phi$, such that $(\lambda, \alpha) \geq 0$. The group U is generated by the U_α

with $(\lambda, \alpha) > 0$. We will also use the Levi subgroup $H \subset P$. Note that $P = H \ltimes U$.

Via $\lambda : \mathbb{G}_m \to G$ the multiplicative group acts by conjugation on G, P and U. We can use this action to twist G, P and U by the $\mathbb{G}_m$-torsor $\mathcal{O}(1)$. We denote the associated twisted groups by G_λ, P_λ and U_λ. For example, $G_\lambda = \mathcal{O}(1) \times_{\mathbb{G}_m, \lambda, Ad} G$.

Proposition 7.3. *We have* $\mathrm{Aut}\, E_\lambda = H \ltimes \Gamma(\mathbb{P}^1, U_\lambda)$.

Proof. We have $\mathrm{Aut}\, E_\lambda = \Gamma(\mathbb{P}^1, G_\lambda) = \Gamma(\mathbb{P}^1, P_\lambda) = H \ltimes \Gamma(\mathbb{P}^1, U_\lambda)$. For more details, see [Ram83], Proposition 5.2. $\qquad\square$

Note that for a semidirect product of linear algebraic groups N, H we have $\mu B(H \ltimes N) = \mu(BH)\mu(BN)$. Thus, we may calculate

$$
\begin{aligned}
\mu(B\,\mathrm{Aut}\, E_\lambda) &= \mu B\big(H \ltimes \Gamma(\mathbb{P}^1, U_\lambda)\big) \\
&= \mu(BH)\mu\big(B\Gamma(\mathbb{P}^1, U_\lambda)\big) \\
&= \frac{\mu(BP)}{\mu(BU)}\mu\big(B\Gamma(\mathbb{P}^1, U_\lambda)\big) \\
&= \mu(BP)\frac{\mu(U)}{\mu\Gamma(\mathbb{P}^1, U_\lambda)} \\
&= \frac{\mu(G/P)}{\mu G}\frac{\mu(U)}{\mu\Gamma(\mathbb{P}^1, U_\lambda)}
\end{aligned}
$$

In the last equation we used the torsor relation for G, i.e., Lemma 2.5.

Now let $\mathfrak{u}$ be the Lie algebra of U. We have $\mathfrak{u} = \bigoplus_{(\lambda,\alpha)>0} \mathfrak{u}_\alpha$, where $\mathfrak{u}_\alpha$ is the Lie algebra of U_α. Since T acts on each $\mathfrak{u}_\alpha$, we obtain line bundles

$$
(\mathfrak{u}_\alpha)_\lambda = \mathcal{O}(1) \times_{\mathbb{G}_m, \lambda} \mathfrak{u}_\alpha \,.
$$

Note that the degree of $(\mathfrak{u}_\alpha)_\lambda$ is equal to (λ, α). The group scheme U_λ over $\mathbb{P}^1$ is a successive extension of the $(\mathfrak{u}_\alpha)_\lambda$, and therefore we have

$$
\mu\Gamma(\mathbb{P}^1, U_\lambda) = \prod_{(\lambda,\alpha)>0} \mathbb{L}^{(\lambda,\alpha)+1} \,,
$$

by Riemann-Roch and hence

$$
\frac{\mu(U)}{\mu\Gamma(\mathbb{P}^1, U_\lambda)} = \mathbb{L}^{-(\lambda, 2\rho)} \,,
$$

where ρ is half the sum of all positive roots.

Denote by $W(\lambda) \subset W$ the subgroup of the Weyl group generated by the reflections coming from simple roots orthogonal to λ. The Bruhat

decomposition for G/P implies

$$\mu(G/P) = \sum_{w \in W/W(\lambda)} \mathbb{L}^{\ell(w)},$$

where $\ell(w)$ is the minimum of all lengths in the coset $wW(\lambda)$.

This finishes the analysis of the motive of $B \operatorname{Aut} E_\lambda$. Putting everything together, we find:

$$\mu(\mathfrak{Bun}_{G,\mathbb{P}^1}) = \sum_{\lambda \in X_*(T)_{\text{dom}}} \mu(B \operatorname{Aut} E_\lambda)$$

$$= \frac{1}{\mu G} \sum_{\lambda \in X_*(T)_{\text{dom}}} \mathbb{L}^{-(\lambda, 2\rho)} \sum_{w \in W/W(\lambda)} \mathbb{L}^{\ell(w)}.$$

The combinatorics of summing the various powers of $\mathbb{L}$ is contained in [KR00]. In fact, it is proved in [ibid.] that

$$\sum_{\lambda \in X_*(T)_{\text{dom}}} \mathbb{L}^{-(\lambda, 2\rho)} \sum_{w \in W/W(\lambda)} \mathbb{L}^{\ell(w)} = |\pi_1(G)| \frac{P(W_{\text{aff}}, \mathbb{L}^{-1})}{P(W, \mathbb{L}^{-1})}.$$

Here the series $P(W_{\text{aff}}, t)$ (resp. $P(W, t)$) is the Poincaré series of the affine Weyl (resp. Weyl) group. It is defined by

$$P(W_{\text{aff}}, t) = \sum_{w \in W_{\text{aff}}} t^{\ell(w)}.$$

It is a result of Bott and Steinberg that

$$\frac{P(W_{\text{aff}}, t)}{P(W, t)} = \prod_{i=1}^{r} (1 - t^{d_i - 1})^{-1}.$$

Thus we may complete the calculation

$$\mu(\mathfrak{Bun}_{G,\mathbb{P}^1}) = |\pi_1(G)| \frac{1}{\mu G} \prod_{i=1}^{r} (1 - \mathbb{L}^{1-d_i})^{-1}$$

$$= |\pi_1(G)| \mathbb{L}^{-\dim G} \prod_{i=1}^{r} (1 - \mathbb{L}^{-d_i})^{-1} \prod_{i=1}^{r} (1 - \mathbb{L}^{1-d_i})^{-1},$$

by Proposition 2.1. In view of the fact that

$$Z(\mathbb{P}^1, u) = (1 - u)^{-1}(1 - \mathbb{L}u)^{-1},$$

this is Conjecture 3.4 for $C = \mathbb{P}^1$.

APPENDIX A. THE MOTIVE OF A TORSOR

By AJNEET DHILLON

A.1. **A Review of Voevodsky's Category of Motives.** We begin by briefly recalling the construction of the triangulated category of effective geometrical motives from [Voe00]. Denote by $\mathrm{SmCor}(k)$ the category whose objects are schemes smooth over k and a morphism from X to Y is an algebraic cycle Z on $X \times Y$ such that each component of Z is finite over X. The finiteness condition allows one to define composition without having to impose an adequate equivalence relation. Note that $\mathrm{SmCor}(k)$ is an additive category with direct sum given by $X \oplus Y = X \amalg Y$.

The homotopy category of bounded of bounded complexes

$$\mathcal{H}^b(\mathrm{SmCor}(k))$$

is a triangulated category. Let T be the minimal thick subcategory containing all complexes of the following two forms:

1) $X \times \mathbb{A}^1 \overset{\mathrm{proj}}{\to} X$

2) For every open cover U, V of X the complex

$$U \cap V \hookrightarrow U \oplus V \overset{(i_u, -i_v)}{\to} X.$$

The triangulated category $\mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}}(k, \mathbb{Z})$ of effective geometrical motives is defined to be the Karoubian hull of the localization of

$$\mathcal{H}^b(\mathrm{SmCor}(k))$$

with respect to T. We will mostly be interested in its $\mathbb{Q}$-linearization $\mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}}(k, \mathbb{Q})$. The obvious functor $\mathrm{Sm}(k) \to \mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}}(k, \mathbb{Q})$ is denoted M_{gm}. We now recall Voevodsky's alternative construction of it.

A presheaf with transfers is a contravariant functor on $\mathrm{SmCor}(k)$. It is called a sheaf with transfers if it is a sheaf when restricted to the big etale site on $\mathrm{Sm}(k)$. We denote by $\mathrm{Shv}(\mathrm{SmCor}(k))$ the category of such sheaves.

A presheaf with transfers F is called homotopy invariant if for all smooth schemes X, the natural map $F(X) \to F(X \times \mathbb{A}^1)$ is an isomorphism. We denote by $\mathrm{DM}_-^{\mathrm{eff}}(k, \mathbb{Z})$ the full subcategory of the derived category $D^-(\mathrm{Shv}(\mathrm{SmCor}(k)))$ consisting of those complexes with homotopy invariant cohomology sheaves. We will be mostly interested in its $\mathbb{Q}$-linearization $\mathrm{DM}_-^{\mathrm{eff}}(k, \mathbb{Q})$.

We denote by $\Delta^\bullet$ the cosimplicial scheme with

$$\Delta^n = \mathrm{Spec}(k[x_0, x_1, \ldots, x_n]/\sum x_i = 1)$$

and face maps given by setting $x_i = 0$. Given a sheaf with transfers F we denote by $C_*(F)$ the complex associated to the simplicial sheaf with transfers whose nth term is

$$C_n(F)(X) = F(X \times \Delta^n).$$

Recall [Voe00, Lemma 3.2.1] that the cohomology sheaves of $C_*(F)$ are homotopy invariant.

Theorem A.1. *The functor $C_*(-)$ extends to a functor*

$$\mathbf{R}C_* : D^-(\mathrm{Shv}(\mathrm{SmCor}(k))) \to \mathrm{DM}_-^{\mathrm{eff}}(k, \mathbb{Q}).$$

This functor is left adjoint to the natural inclusion.

Proof. See [Voe00, Theorem 3.2.3,§3.3]. $\qquad\qquad\qquad\qquad\square$

Theorem A.2. *For a perfect field k there is a commutative diagram of functors*

$$
\begin{array}{ccc}
\mathcal{H}^b(\mathrm{SmCor}(k)) \otimes \mathbb{Q} & \xrightarrow{\;L\;} & D^-(\mathrm{Shv}(\mathrm{SmCor}(k))) \otimes \mathbb{Q} \\
\downarrow & & \downarrow \\
\mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}}(k, \mathbb{Q}) & \xrightarrow{\quad i \quad} & \mathrm{DM}_-^{\mathrm{eff}}(k, \mathbb{Q})
\end{array}
$$

such that i is a fully faithful embedding with dense image.

Proof. See [Voe00, Theorem 3.2.6] including the construction of i. $\quad\square$

If the field k admits a resolution of singularities then there is a functor called the motive with compact support:

$$M_{gm}^c : \mathrm{sch}^{prop}/k \to \mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}}(k, \mathbb{Q}),$$

here sch^{prop}/k is the category whose objects are schemes of finite type over k and morphisms are proper maps. If Z is a closed subscheme of X then there is an exact triangle

$$M_{gm}^c(Z) \to M_{gm}^c(X) \to M_{gm}^c(X - Z) \to M_{gm}^c(Z)[1].$$

For further properties see [Voe00, pg. 195]. We will now briefly recall the construction of M_{gm}^c. For a scheme X and a smooth scheme U define $L^c(X)(U)$ to be the free abelian group generated by closed integral subschemes of $X \times U$ quasi-finite over U and dominant over a component of U. In this way we obtain a sheaf

$$L^c(X) : \mathrm{SmCor}(k)^{op} \to \mathbf{Ab}.$$

We have a functor

$$L^c : \mathrm{sch}^{prop}/k \to \mathrm{Shv}(\mathrm{SmCor}(k)).$$

The motive with compact supports of X is defined to be $\mathbf{R}\mathrm{C}_* L^c(X)$. It is a theorem, see [Voe00, Corollary 4.1.4] that this sheaf belongs to $\mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}(k, \mathbb{Q})$.

Proposition A.3. *Let Γ be a finite group acting on the scheme X with quotient $Y = X/\Gamma$. Then*

$$L^c(X) \to L^c(Y)$$

is a quotient in $\mathrm{Shv}(\mathrm{SmCor}(k)) \otimes \mathbb{Q}$.

Proof. The category $\mathrm{Shv}(\mathrm{SmCor}(k)) \otimes \mathbb{Q}$ is equivalent to the category of sheaves of $\mathbb{Q}$-vector spaces with the equivalence being given by the functor $- \otimes \mathbb{Q}$. For every U the natural push forward map induces an isomorphism

$$L^c(X)(U)^\Gamma \to L^c(Y)(U),$$

see [Ful98, 1.7.6]. The result now follows. $\qquad\qquad\square$

Corollary A.4. *In the above notation the natural map*

$$L^c(X) \to L^c(Y)$$

is a quotient in $D^-(\mathrm{Shv}(\mathrm{SmCor}(k))) \otimes \mathbb{Q}$.

Proof. Note that one can calculate maps from $L^c(X)$ to $\mathcal{F}$ in the derived category by taking an injective resolution $I_\bullet$ of $\mathcal{F}$ and calculating homotopy classes of maps to $I_\bullet$. So suppose

$$\phi : L^c(X) \to I_\bullet$$

is Γ-equivariant. So for each $\gamma \in \Gamma$ there exists

$$h_\gamma : L^c(X) \to I_1$$

with $\phi \circ \gamma - \phi = d_1 \circ h_\gamma$. Then $\frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} \phi \circ \gamma$ is Γ equivariant in $\mathrm{Shv}(\mathrm{SmCor}(k)) \otimes \mathbb{Q}$ and this same map equals ϕ in the homotopy category. Now apply the proposition. $\qquad\qquad\square$

Corollary A.5. *In the same notation*

$$M^c_{gm}(X) \to M^c_{gm}(Y)$$

is a quotient in $\mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}(k, \mathbb{Q})$.

Proof. This is because $\mathbf{R}\mathrm{C}_*$ is left adjoint to the inclusion. $\qquad\square$

Proposition A.6. *Consider the family of inclusions*

$$i_s : X \hookrightarrow X \times \mathbb{A}^1.$$

Then $\mathrm{C}_* L^c(i_1) = \mathrm{C}_* L^c(i_0)$ in $\mathcal{H}^b(\mathrm{Shv}(\mathrm{SmCor}(k)) \otimes \mathbb{Q})$.

Proof. This is well known. A proof can be found in [MVW, Lemma 2.17]. $\qquad\qquad\square$

A.2. **The Main Result.** We assume throughout this section the characteristic of the ground field k is 0.

Theorem A.7. *Let G be a split semisimple connected group or a connected unipotent group over k. Let $x \in G(A)$ where A is a finite generated k algebra that is a domain. Then there is an open affine $\mathrm{Spec}(A') \subseteq \mathrm{Spec}(A)$ and a finite Galois cover $\mathrm{Spec}(B) \to \mathrm{Spec}(A')$ and $y \in G(B[t])$ such that*
(i) $y(0)$ is the constant morphism to the identity
(ii) The following diagram commutes

$$
\begin{array}{ccc}
\mathrm{Spec}(A') & \longrightarrow & \mathrm{Spec}A \xrightarrow{\;x\;} G \\
\uparrow & & \nearrow {\scriptstyle y(1)} \\
\mathrm{Spec}B & &
\end{array}
$$

Proof. The result is straightforward in the case where G is a connected unipotent group as in this case the underlying variety of G is an $\mathbb{A}^n$, see [Spr98, pg. 243].

So we assume G is semisimple. Let $\tilde{G}$ be the universal cover of G. As

$$\tilde{G} \to G$$

is Galois, by replacing $\mathrm{Spec}(A)$ by $\mathrm{Spec}(A) \times_{G,x} \tilde{G}$ we may assume that G is simply connected.

According to [Ste62] there is a unipotent group U and a morphism

$$\phi : U \to G$$

that is surjective on L points for every field L. Let K be the function field of A and x_K be the A-point of G restricted to K. There is a K-point x' of U mapping to x_K via ϕ. By examining denominators we can find an open affine

$$\mathrm{Spec}(A') \hookrightarrow \mathrm{Spec}(A)$$

such that $x_{A'}$ lifts to a A'-point z of U. Now the underlying variety of U is again an $\mathbb{A}^n$. $\qquad\square$

Let $\pi : Y \to X$ be a finite Galois cover with Galois group Γ. Let P be a G-torsor trivialized by Y, that is $Y \times_X P \xrightarrow{\sim} Y \times G$. The action of Γ on Y lifts to an action of Γ on $Y \times G$ with quotient P. This action is determined by a 1-cocyle

$$n : \Gamma \times Y \to G.$$

Proposition A.8. *Let G be a connected split semisimple group or a group whose underlying variety is $\mathbb{A}^n$. Let $P \to X$ be a G-torsor. Then there is an open affine $X' \subseteq X$ and a Galois cover $Y' \to X'$ trivializing P. Furthermore if Γ is the Galois group and the cocycle*

$$n : \Gamma \times Y \to G$$

defines the action we may assume that n extends to

$$n_t : \Gamma \times Y \times \mathbb{A}^1 \to G$$

with $n_0 = n$ and n_1 constant at the identity.

Proof. The first part is standard, using Zariski's main theorem. The second part is by repeated applications of A.7.   □

We denote by $K_0(\mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}(k, \mathbb{Q}))$ the K-group of the triangulated category $\mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}(k, \mathbb{Q})$. It is the free abelian group on the objects of $\mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}(k, \mathbb{Q})$ subject to the relations

$$Y = X + Z \quad \text{for each exact triangle} \quad X \to Y \to Z \to X[1].$$

The tensor product of the category $\mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}(k, \mathbb{Q})$ makes $K_0(\mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}(k, \mathbb{Q}))$ into a ring. We have a ring homomorphism

$$\chi^c_{\mathrm{Mot}} : K_0(Var_k) \to K_0(\mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}(k, \mathbb{Q}))$$

given by the motive with compact supports. We denote by $\chi^c_{\mathrm{Mot}}(X)$ the image of the variety X under this homomorphism.

Theorem A.9. *Let G be a connected linear algebraic group that is split over k. Let P be a G-torsor over X with X of finite type. Then $\chi^c_{\mathrm{Mot}}(P) = \chi^c_{\mathrm{Mot}}(X)\chi^c_{\mathrm{Mot}}(G)$.*

Proof. By noetherian induction it suffices to find an open subset X' of X such that

$$\chi^c_{\mathrm{Mot}}(X')\chi^c_{\mathrm{Mot}}(G) = \chi^c_{\mathrm{Mot}}(P|_U).$$

First we assume that G is as in A.8. Then we can find Γ, n_t X' and Y' as in the proposition. Now the natural map

$$M^c_g m(P \times_X Y') \to M^c_{gm}(P|_{X'})$$

is a quotient by A.5. On the other hand by A.6 the cocycle is $M^c_{gm}(n)$ is trivial. Hence the result.

For a general group note that $P \to P/R_u(G)$ is a $R_u(G)$-torsor. So we may assume G is split reductive. But then $R(G)$ is a torus and every torsor for a torus is Zariski trivial so we are reduced to the semisimple case.   □

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