On the Cohomology of Stable Map Spaces

K. Behrend and A. O'Halloran

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Abstract

We describe an approach to calculating the cohomology rings of stable map spaces $\overline{M}_{0,0}(\mathbb{P}^n, d)$.

Contents

Introduction

Spaces of stable maps have enjoyed a lot of interest in recent years. They were first introduced by Kontsevich in 1994 (see [13]), and have since proven to be very useful in many contexts, especially in Quantum Cohomology and Mirror Symmetry. Stable map spaces are natural completions of spaces of morphisms from algebraic curves to a fixed (non-singular complete) variety X. They arise as natural generalizations of the moduli spaces of stable curves discovered by Deligne and Mumford [8].

Here we shall be mostly concerned with the spaces

$$\overline{M}_{0,0}(\mathbb{P}^n, d) \,.$$

The generic member of $\overline{M}_{0,0}(\mathbb{P}^n, d)$ (at least if $n \geq 3$) is a non-singular rational curve of degree d in projective n-space and, in fact, $\overline{M}_{0,0}(\mathbb{P}^n, d)$ is a compactification of the space of all such curves, and thus has dimension $dn + d + n - 3$. The degenerations we allow at the boundary are pairs (C, f), where C is a nodal curve of arithmetic genus 0 and f is a morphism $f : C \to \mathbb{P}^n$ of degree d, such that every component of C which is contracted to a point by f has at least 3 nodes. This also gives the correct picture for $n = 1, 2$. For example, $\overline{M}_{0,0}(\mathbb{P}^1, d)$ compactifies the space of degree d ramified covers of genus zero of the projective line. We also remark that $\overline{M}_{0,0}(\mathbb{P}^n, 1)$ is simply the Grassmannian $G(2, n + 1)$ of lines in $\mathbb{P}^n$.

The true beauty of the spaces $\overline{M}_{0,0}(\mathbb{P}^n, d)$ only becomes apparent if we consider them as *stacks*. In fact, the algebraic stacks underlying the various $\overline{M}_{0,0}(\mathbb{P}^n, d)$ are smooth and admit universal families. These are properties that the spaces $\overline{M}_{0,0}(\mathbb{P}^n, d)$ generally lack. We shall always work with these stacks and thus use the notation $\overline{M}_{0,0}(\mathbb{P}^n, d)$ for the *stack of stable maps* of degree d to $\mathbb{P}^n$ (of genus 0 without marked points).

Our goal is to compute the cohomology ring of $\overline{M}_{0,0}(\mathbb{P}^n, d)$, or at least, to outline an approach by which this might be achieved. (The only previous result in this direction is the computation of the Betti numbers of $\overline{M}_{0,\nu}(\mathbb{P}^n, d)$ due to Getzler and Pandharipande [9].) Our method is inspired by the utility of $\mathbb{C}^*$-actions for studying integrals over stable map spaces, but there is an additional ingredient: a vector field (which is compatible with the $\mathbb{C}^*$-action).

This method is due to Akildiz and Carrell [1] and can be summarized as follows. Let X be a non-singular projective variety over $\mathbb{C}$ with a $\mathbb{C}^*$-action and suppose that V is a vector field on X, satisfying $^\lambda V = \lambda V$, for all $\lambda \in \mathbb{C}^*$ (we say that V is *equivariant*). If V has exactly one fixed point and Z is the scheme-theoretic fixed locus of V (so Z is a one-point non-reduced scheme), then we have

$$H^*(X, \mathbb{C}) = \Gamma(Z, \mathcal{O}_Z) \,.$$

(See Examples 1.3 and 1.7, below, where this is worked out for the special case of $X = \mathbb{P}^n$. See also Remark 4.30, for the case of the Grassmannian of lines in $\mathbb{P}^n$.)

Our use of the method of Akildiz-Carrell is novel in two aspects: we apply it to stacks, but more significantly, the fixed locus Z of the vector field V has positive dimension. Thus we have to replace the ring of global sections $\Gamma(Z, \mathcal{O}_Z)$ by the hypercohomology ring $\mathbb{H}^0(X, K_V^\bullet)$, where $K_V^\bullet$ is the Koszul complex defined by the vector field V.

There is one important difference between $\Gamma(Z, \mathcal{O}_Z)$ and $\mathbb{H}^0(X, K_V^\bullet)$. The ring $\Gamma(Z, \mathcal{O}_Z)$ can be computed entirely on the fixed locus Z, whereas $\mathbb{H}^0(X, K_V^\bullet)$ depends on an open neighborhood of Z in X. Thus, in the case of positive-dimensional fixed locus Z, the localizing power of the method is much weaker.

The method is saved by a somewhat surprising phenomenon. We discovered that we can restrict our attention entirely to a certain open subset U of X, even though U does *not* cover the fixed locus Z completely. This open subset U is the 'big cell' associated by Białynicki-Birula [5] to the $\mathbb{C}^*$-action on X.

In the cases we consider here, it turns out that the canonical map

$$H^*(X, \mathbb{C}) = \mathbb{H}^0(X, K_V^\bullet) \longrightarrow \mathbb{H}^0(U, K_V^\bullet) = \Gamma(U, \mathcal{O}_Z)$$

is, though not injective, injective in all *relevant* degrees. This means injective in all degrees that contain a generator or a relation.

One of our main results is an explicit description of the big Białynicki-Birula cell of $\overline{M}_{0,0}(\mathbb{P}^n, d)$ as a vector bundle over $\overline{M}_{0,d}$ (modulo an action of the symmetric group S_d). Here $\overline{M}_{0,d}$ is the space of stable curves of genus zero with d marked points, which is comparatively well understood.

The case $d = 3$ is particularly simple and we focus on it in the latter part of the paper. If $d = 3$ then $\overline{M}_{0,d}$ is just a point and so the Białynicki-Birula cell is simply an affine space $\mathbb{A}^{4n}$ modulo an action of S_3. We succeed in writing down the vector field V in canonical coordinates on $\mathbb{A}^{4n}$. This leads at least to a conjectural description of the cohomology ring of $\overline{M}_{0,0}(\mathbb{P}^n, 3)$. The truth of this conjecture depends only on a certain purely algebraic statement, which we verified using Macaulay 2 [10] for $n \leq 5$.

More interesting than the case of finite n is the case of the limit as n approaches ∞. The cohomology ring of $\overline{M}_{0,0}(\mathbb{P}^n, d)$ stabilizes as n increases, so we can define a ring which we call the cohomology ring of $\overline{M}_{0,0}(\mathbb{P}^\infty, d)$, even though this latter stack does not make sense.

We succeed in describing the cohomology ring of $\overline{M}_{0,0}(\mathbb{P}^\infty, 3)$ completely using generators and relations (Theorem 4.15). This is the main result of the paper. It says

$$H^*\big(\overline{M}_{0,0}(\mathbb{P}^\infty, 3), \mathbb{C}\big) = \mathbb{C}[b, \sigma_1, \rho, \sigma_2, \tau, \sigma_3]/\big((\tau^2 - \rho\sigma_2), \tau\sigma_3, \rho\sigma_3\big).$$

The generators can be expressed in terms of Chern classes of certain canonical vector bundles on $\overline{M}_{0,0}(\mathbb{P}^\infty, 3)$. The degrees of b and σ_1 are 1, the degrees of ρ, σ_2 and τ are 2 and the degree of σ_3 is 3, using algebraic degrees (where the first Chern class has degree 1). Thus the degrees of the relations are 4, 5 and 5, respectively.

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Thus, as a ring, $H_{DR}\big(\overline{M}_{0,0}(\mathbb{P}^\infty, 3)\big)$ is reduced, of pure dimension 4 and has two irreducible components, $\mathbb{C}[b, \sigma_1, \sigma_2, \sigma_3]$ and $\mathbb{C}[b, \sigma_1, \rho, \sigma_2, \tau]/(\tau^2 - \rho\sigma_2)$, intersecting transversally along $\mathbb{C}[b, \sigma_1, \sigma_2]$.

The case $d = 2$ is special, as $\overline{M}_{0,2}$ is not defined. It is much easier than the case $d = 3$ and we have complete results.

We briefly outline the structure of the paper.

In Section 1 we describe the theory of equivariant vector fields and their relation to de Rham cohomology. We verify that the results of Akildiz and Carrell which we require hold for stacks. We improve on existing treatments of Chern classes by proving that the Carrell-Lieberman class [7] is *homogeneous* (see Section 1.3). Hence the Carrell-Lieberman characteristic classes (and not just their leading terms) are equal to the corresponding Chern classes.

Section 2 assembles a few facts about stable map stacks which we require later. We observe that if E is a convex vector bundle on the variety X, then the stack of stable maps to E is a vector bundle over the stack of stable maps to X. We prove that the cohomology of $\overline{M}_{0,\nu}(\mathbb{P}^n, d)$ stabilized as n increases and we define the cohomology ring of $\overline{M}_{0,\nu}(\mathbb{P}^\infty, d)$. Moreover, this cohomology ring maps surjectively onto the cohomology ring of $\overline{M}_{0,\nu}(\mathbb{P}^n, d)$, for every finite n.

In Section 3 we describe the big Białynicki-Birula cell of $\overline{M}_{0,0}(\mathbb{P}^n, d)$ as a vector bundle over $[\overline{M}_{0,d}/S_d]$. The most significant case is $n = 1$. Here the big Białynicki-Birula cell consists of all stable maps which are unramified over $\infty \in \mathbb{P}^1$. In the general case, it consists of all stable maps which avoid the codimension 2 plane $\langle 0, 0, *, \ldots, * \rangle$ and intersect the hyperplane $\langle 0, *, \ldots, * \rangle$ transversally d times.

By changing the $\mathbb{C}^*$-action on $\mathbb{P}^n$, we can cover all of $\overline{M}_{0,0}(\mathbb{P}^n, d)$ with vector bundles over $\overline{M}_{0,d}$, and so our results lead, at least in principle, to an explicit description of the stable map stacks $\overline{M}_{0,0}(\mathbb{P}^n, d)$ in terms of stable curve spaces $\overline{M}_{0,d}$.

Section 4 starts with a recipe to calculate our vector field on $\overline{M}_{0,0}(\mathbb{P}^n, d)$. The key result is that the derivative of the universal map $f : \widetilde{C} \to \mathbb{P}^n$ induces an isomorphism $\Gamma(\widetilde{C}, \mathcal{T}_{\widetilde{C}}) \to \Gamma(\widetilde{C}, f^*\mathcal{T}_{\mathbb{P}^n})$. The remainder of the section contains the calculations for the cases $d = 2$ and $d = 3$.

Notation and Conventions

Throughout the paper we will work over the ground field $\mathbb{C}$ of complex numbers.

All of our algebraic stacks will be of Deligne-Mumford type. This means that the diagonal $X \to X \times X$ is unramified. Deligne-Mumford stacks X admit étale presentations $U \to X$, where U is a scheme. We denote the stack quotient associated to a G-variety X by $[X/G]$.

Whenever we consider sheaves on a Deligne-Mumford stack X, it is understood that these are sheaves on the small étale site of X. Objects of this étale site are thus étale morphisms $U \to X$, where U is a scheme. The topology on this site is defined in the same way as for the étale site of a scheme. Any vector bundle $E \to X$ defines a sheaf of local sections on the étale site of X, which we often identify with E.

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Any cohomology group of a sheaf on X is understood to be the cohomology of the étale site with values in the given sheaf, unless mentioned otherwise.

Stable maps

For an algebraic variety X (not necessarily proper), we denote by $H_2(X)^+$ the semigroup (with 0) of group homomorphisms $\mathrm{Pic}(X) \to \mathbb{Z}$, which take non-negative values on ample line bundles. This semigroup is a convenient set of labels for the class of a stable map. Given a stable map (C, x, f) to X, it is of class $\beta \in H_2(X)^+$ if $\deg(f^*L) = \beta(L)$, for all $L \in \mathrm{Pic}(X)$. If $X = \mathbb{P}^n$, we identify $H_2(X)^+$ with $\mathbb{Z}_{\geq 0}$.

We denote by $\overline{M}_{g,n}(X, \beta)$ the stack of stable maps of class β from n-marked genus g curves to X.

Group actions

If an algebraic group G acts on a smooth scheme or a smooth algebraic stack X, we will denote this action on the right. Assume given a right action of G on X and a lift of this action to a vector bundle E over X. Then for any $g \in G$ and any local section $e \in \Gamma(Ug, E)$, we denote by ${}^g e$ the section of E over U given by the formula

$$({}^g e)(x) = e(xg)g^{-1} \,. \tag{1}$$

In other words, ${}^g e \in \Gamma(U, E)$ is defined to make the diagram

$$
\begin{array}{ccc}
E & \xrightarrow{\ g\ } & E \\
{\scriptstyle {}^g e}\big\uparrow & & \big\uparrow{\scriptstyle e} \\
U & \xrightarrow{\ g\ } & Ug
\end{array}
$$

commute. In particular, (1) defines a left representation of G on the $\mathbb{C}$-vector space $\Gamma(X, E)$ of global sections of E.

Any action of G on X lifts naturally to the vector bundles $\mathcal{O}_X$, $\mathcal{T}_X$ and Ω_X, and so we get induced representations of G on functions, vector fields and differential forms on X. Explicitly, if $f \in \Gamma(X, \mathcal{O}_X)$ is a regular function on X, then ${}^g f = g^* f$, or $({}^g f)(x) = f(xg)$, for all $x \in X$. If $V \in \Gamma(X, \mathcal{T}_X)$ is a vector field on X, then ${}^g V$ is characterized by the formula

$$Dg(x)\big(({}^g V)(x)\big) = V(xg)$$

or more briefly by

$$(Dg)({}^g V) = g^* V \,.$$

(Here $Dg : \mathcal{T}_X \to g^* \mathcal{T}_X$ is the derivative of $g : X \to X$.) If $\omega \in \Gamma(X, \Omega)$ is a differential form on X, then ${}^g \omega$ is given by

$$ {}^g \omega = dg(g^* \omega) \,,$$

or
$$({}^g\omega)(x) = dg(x)\big(\omega(xg)\big),$$

where $dg : g^*\Omega_X \to \Omega_X$ denotes the natural pullback homomorphism. Note that for all $g \in G$ we have

$$ {}^g\langle \omega, V \rangle = \langle {}^g\omega, {}^gV \rangle. \tag{2}$$

If F is another vector bundle over X to which the G-action has been lifted, then we get induced G-actions also on the vector bundles $\mathcal{H}om(E, F)$ and $E \otimes F$, given by the formulas $(\phi g)(eg) = \phi(e)g$ and $(e \otimes f)g = eg \otimes fg$. On global sections, this gives rise to G-representations by the formulas

$$({}^g\phi)({}^ge) = {}^g\big(\phi(e)\big)$$

and

$$ {}^g(e \otimes f) = {}^ge \otimes {}^gf. $$

Finally, given a $\mathbb{C}$-linear sheaf homomorphism $\Phi : E \to F$, we define ${}^g\Phi$ by

$$({}^g\Phi)({}^ge) = {}^g\big(\Phi(e)\big),$$

for any local section e of E. This generalizes the definitions above if Φ is $\mathcal{O}_X$-linear and gives rise to a G-representation on the space of all $\mathbb{C}$-linear sheaf homomorphisms from E to F. For example, the universal derivation $d : \mathcal{O}_X \to \Omega_X$ satisfies ${}^gd = d$, for all $g \in G$ (this follows easily from (2)). Note that, because of this, ${}^g\nabla$, for a connection ∇ on E, is again a connection on E.

Whenever any kind of object A satisfies an equation ${}^gA = A$, for all $g \in G$, then we refer to A as *G-invariant*.

The case of $\mathbb{G}_m$

In many cases our group G will be equal to the multiplicative group $\mathbb{G}_m$. If we are given a right $\mathbb{G}_m$-action on a vector bundle E, lifting a right action of $\mathbb{G}_m$ on X, then we refer to this action as the *geometric* action, to distinguish it from the action of $\mathbb{G}_m$ on E by scalar multiplication on the fibers, which we shall call the *linear* action. Of course the geometric and the linear action commute with each other.

In this case the (geometric) $\mathbb{G}_m$-action gives rise to a $\mathbb{C}^*$-representation on all the above mentioned vector spaces, i.e., it makes them into graded $\mathbb{C}$-vector spaces. The homogeneous elements A of degree i satisfy

$$ {}^\lambda A = \lambda^i A, $$

for all $\lambda \in \mathbb{C}^*$. Of particular interest to us are elements of degree one; we shall call them *$\mathbb{G}_m$-equivariant*, or just *equivariant*, since there is rarely any fear of confusion. In particular, this gives rise to the notion of *equivariant vector field*.

1 Outline of the Method

Let X be a smooth and proper Deligne-Mumford stack. Assume that

$$H^p(X, \Omega^q) = 0, \text{ for all } p \neq q,$$

where $\Omega = \Omega_X$ is the sheaf of Kähler differentials on X. (See Remark 1.8.)

We will be interested in the graded ring

$$H_{DR}(X) = \bigoplus_p H^p(X, \Omega^p) \ .$$

The notation $H_{DR}(X)$ is justified, because under our assumption

$$\bigoplus_p H^p(X, \Omega^p) = \mathbb{H}\big(X, (\Omega^\bullet, d)\big) \ ,$$

where $(\Omega^\bullet, d)$ is the algebraic de Rham complex of X. We use the algebraic grading, i.e., we consider $H^p(X, \Omega^p)$ to have degree p.

Remark 1.1 Let X_{an} be the (small) analytic site of X. By a theorem of Grothendieck (see [11]) we have

$$\mathbb{H}\big(U, (\Omega^\bullet, d)\big) = H(U_{\mathrm{an}}, \mathbb{C}),$$

for every smooth variety U. Choosing an étale presentation $U \to X$, where U is a smooth variety, we obtain E_2-spectral sequences

$$\mathbb{H}^q\big(U_p, (\Omega^\bullet, d)\big) \Rightarrow \mathbb{H}^{p+q}\big(X, (\Omega^\bullet, d)\big)$$

and

$$H^q(U_{p_{\mathrm{an}}}, \mathbb{C}) \Rightarrow H^{p+q}(X_{\mathrm{an}}, \mathbb{C}) \, ,$$

where U_p is the $(p+1)$-fold fibered product of U with itself over X.

Thus we conclude that

$$\mathbb{H}\big(X, (\Omega^\bullet, d)\big) = H(X_{\mathrm{an}}, \mathbb{C}) \ .$$

Letting $\overline{X}$ be the coarse moduli space of X, we have

$$H(X_{\mathrm{an}}, \mathbb{C}) = H(\overline{X}_{\mathrm{an}}, \mathbb{C}) \ ,$$

essentially because group cohomology of any finite group with values in $\mathbb{C}$ vanishes. Thus we have

$$H_{DR}(X) = H(X_{\mathrm{an}}, \mathbb{C}) = H(\overline{X}_{\mathrm{an}}, \mathbb{C}) \ ,$$

and so we can also interpret our ring $H_{DR}(X)$ as the usual (singular) cohomology ring of the topological space underlying the variety $\overline{X}$ over $\mathbb{C}$.

Note that for $H(X_{\mathrm{an}}, \mathbb{C})$ we also use the algebraic grading, i.e., we consider $H^{2p}(X_{\mathrm{an}}, \mathbb{C})$ to have degree p.

1.1 Equivariant vector fields

Now assume that we are given a (right) $\mathbb{G}_{\mathrm{m}}$-action on X. Recall that we defined a vector field V on X to be $\mathbb{G}_{\mathrm{m}}$-equivariant if ${}^{\lambda}V = \lambda V$, for all $\lambda \in \mathbb{C}^*$. Note that V is equivariant if and only if the diagram

$$
\begin{array}{ccc}
\lambda^* \Omega_X & \xrightarrow{\;\lambda^* V\;} & \lambda^* \mathcal{O}_X \\[2pt]
{\scriptstyle \lambda d\lambda} \big\downarrow & & \big\downarrow {\scriptstyle =} \\[2pt]
\Omega_X & \xrightarrow{\;\;V\;\;} & \mathcal{O}_X
\end{array}
$$

commutes, for all $\lambda \in \mathbb{C}^*$.

Remark 1.2 Let G be the semidirect product of the multiplicative group and the additive group, where the action of $\mathbb{G}_{\mathrm{m}}$ by conjugation on $\mathbb{G}_{\mathrm{a}}$ is given by scalar multiplication: ${}^{\lambda}a = \lambda a$. We can identify G with the group of 2×2 invertible matrices of the form

$$
\begin{pmatrix} 1 & 0 \\ a & \lambda \end{pmatrix} .
$$

Suppose we are given a right action of G on X. Then restricting to the multiplicative subgroup of G gives us a (right) action of $\mathbb{G}_{\mathrm{m}}$ on X. Taking the derivative of the $\mathbb{G}_{\mathrm{a}}$-action on X defines a vector field V on X. More precisely, let

$$
\phi_x : \mathbb{A}^1 \longrightarrow X
$$
$$
a \longmapsto xa
$$

denote the orbit map of $x \in X$ for the action of $\mathbb{G}_{\mathrm{a}}$. Then $V(x) = D\phi_x(0)$, where we identify the linear map

$$
D\phi_x(0) : \mathcal{T}_{\mathbb{A}^1}(0) \longrightarrow \mathcal{T}_X(x)
$$

with the image of the canonical generator $1 \in \mathcal{T}_{\mathbb{A}^1}(0)$.

Taking the derivative at 0 of the commutative diagram

$$
\begin{array}{ccc}
\mathbb{A}^1 & \xrightarrow{\;\phi_{x\lambda}\;} & X \\[2pt]
{\scriptstyle \lambda} \big\downarrow & & \big\uparrow {\scriptstyle \lambda} \\[2pt]
\mathbb{A}^1 & \xrightarrow{\;\phi_x\;} & X
\end{array}
$$

proves that $D\phi_{x\lambda}(0) = D\lambda(x)D\phi_x(0)\lambda$, and hence that ${}^{\lambda}V(x) = \lambda V(x)$. Thus V is $\mathbb{G}_{\mathrm{m}}$-equivariant. This is the most common source of $\mathbb{G}_{\mathrm{m}}$-equivariant vector fields.

Example 1.3 Let D_λ denote the diagonal $(n+1) \times (n+1)$-matrix with entries $(1, \lambda, \ldots, \lambda^n)$ along the diagonal. Let N denote the nilpotent $(n+1) \times (n+1)$-matrix with ones along the sub-diagonal and zeros elsewhere. Then

$$\begin{pmatrix} 1 & 0 \\ a & \lambda \end{pmatrix} \longmapsto e^{aN} D_\lambda$$

defines a group homomorphism $G \to GL(n+1)$. Via this homomorphism, we define a right action of G on $\mathbb{P}^n$, by acting in the natural way on homogeneous coordinate vectors of $\mathbb{P}^n$, which we consider to be row vectors of length $n+1$.

As in Remark 1.2, we get an induced $\mathbb{G}_m$-action and an equivariant vector field W on $\mathbb{P}^n$. In standard homogeneous coordinates on $\mathbb{P}^n$ this vector field is given by

$$W = \sum_{i=1}^n x_i \frac{\partial}{\partial x_{i-1}} \, .$$

The zero locus of this vector field consists of one point, namely the origin in the standard affine space $\mathbb{A}^n \subset \mathbb{P}^n$ defined by $x_0 = 1$. In affine coordinates $s_i = \frac{x_i}{x_0}$ this vector field is given by

$$W = -s_1 s_n \frac{\partial}{\partial s_n} + \sum_{i=1}^{n-1} (s_{i+1} - s_1 s_i) \frac{\partial}{\partial s_i} \, .$$

(Recall the relation $\sum_{i=0}^n x_i \frac{\partial}{\partial x_i} = 0$.)

The Koszul complex

Let X be as above, endowed with a $\mathbb{G}_m$-action. Let $V : \Omega_X \to \mathcal{O}_X$ be an equivariant vector field on X. We can associate to V the Koszul complex

$$\Omega_X^N \xrightarrow{\iota(V)} \ldots \xrightarrow{\iota(V)} \Omega_X^2 \xrightarrow{\iota(V)} \Omega_X \xrightarrow{V} \mathcal{O}_X \, ,$$

where N denotes the dimension of X, hence the rank of Ω_X, and $\iota(V)$ denotes contraction with V. (If we pull back this Koszul complex to an étale and affine X-scheme over which we can trivialize Ω_X, then the vector field V is given by N regular functions and the above complex is the usual Koszul complex associated to this sequence of regular functions.)

We set $K^p = \Omega_X^{-p}$, for $p \in \mathbb{Z}$, and denote the above Koszul complex by

$$K_V^\bullet = \big(K^\bullet, \iota(V) \big) \, .$$

Note that $K_V^\bullet$ is a sheaf of differential graded commutative (with unit) $\mathcal{O}_X$-algebras on X.

We are interested in the hypercohomology $\mathbb{H}(X, K_V^\bullet)$, and its relation to $H_{DR}(X)$.

Lemma 1.4 *For every $p \neq 0$ we have $\mathbb{H}^p(X, K_V^\bullet) = 0$. Moreover, $\mathbb{H}^0(X, K_V^\bullet)$ is a commutative filtered $\mathbb{C}$-algebra, and for the associated graded algebra we have*

$$\operatorname{gr} \mathbb{H}^0(X, K_V^\bullet) = H_{DR}(X) \,.$$

PROOF. This follows immediately from the standard E_1 spectral sequence of hypercohomology and our assumption on the vanishing of off-diagonal Hodge groups. $\square$

For $\lambda \in \mathbb{C}^*$ we define a homomorphism $\psi_\lambda : \lambda^* K_V^\bullet \to K_V^\bullet$ by

$$
\begin{array}{ccccccccc}
\cdots & \longrightarrow & \lambda^* \Omega_X^p & \xrightarrow{\lambda^* \iota(V)} & \cdots & \longrightarrow & \lambda^* \Omega_X & \xrightarrow{\lambda^* V} & \lambda^* \mathcal{O}_X \\
& & {\scriptstyle \lambda^p d\lambda} \big\downarrow & & & & {\scriptstyle \lambda d\lambda} \big\downarrow & & \big\downarrow {\scriptstyle =} \\
\cdots & \longrightarrow & \Omega_X^p & \xrightarrow{\iota(V)} & \cdots & \longrightarrow & \Omega_X & \xrightarrow{V} & \mathcal{O}_X
\end{array}
\tag{3}
$$

Here $d\lambda : \lambda^* \Omega_X^p \to \Omega_X^p$ is the homomorphism induced on the exterior power by the derivative $d\lambda : \lambda^* \Omega_X \to \Omega_X$. Note that ψ_λ is a homomorphism of differential graded algebras.

We can use ψ_λ to define a representation of $\mathbb{C}^*$ on the hypercohomology $\mathbb{H}^0(X, K_V^\bullet)$. We simply associate to $\lambda \in \mathbb{C}^*$ the automorphism

$$\mathbb{H}^0(X, K_V^\bullet) \xrightarrow{\lambda^*} \mathbb{H}^0(X, \lambda^* K_V^\bullet) \xrightarrow{\psi_\lambda} \mathbb{H}^0(X, K_V^\bullet) \,.$$

Thus $\mathbb{H}^0(X, K_V^\bullet)$ becomes a graded $\mathbb{C}$-algebra.

Proposition 1.5 (Akildiz-Carrell [1]) *There is a canonical isomorphism of graded $\mathbb{C}$-algebras*

$$\mathbb{H}^0(X, K_V^\bullet) = H_{DR}(X) \,.$$

PROOF. Note that the grading induced by the $\mathbb{C}^*$-representation on $\mathbb{H}^0(X, K_V^\bullet)$ is compatible with the filtration induced by the E_1-spectral sequence. Thus we get an induced $\mathbb{C}^*$-representation on the associated graded algebra. One shows that via this induced representation, $\mathbb{C}^*$ acts on $H^p(X, \Omega^p)$ through the character $\lambda \mapsto \lambda^p$. Then the proof is finished, in view of the lemma from linear algebra which we state below.

For the convenience of the reader, we recall the proof that $\mathbb{C}^*$ acts on $H^p(X, \Omega^p)$ through $\lambda \mapsto \lambda^p$. One simply factors ψ_λ as $\psi_\lambda = \phi_\lambda \circ d\lambda$, where ϕ_λ is multiplication by λ^p on Ω^p. Thus the action of $\lambda \in \mathbb{C}^*$ on $H^p(X, \Omega^p)$ factors as $\psi_\lambda \circ \lambda^* = \phi_\lambda \circ (d\lambda \circ \lambda^*)$. Now $d\lambda \circ \lambda^*$ is the homomorphism on de Rham cohomology induced by the morphism $\lambda : X \to X$, which is the identity. On the other hand, ϕ_λ obviously induces multiplication by λ^p on $H^p(X, \Omega^p)$. $\square$

Lemma 1.6 *Let H be a commutative filtered $\mathbb{C}$-algebra with a $\mathbb{C}^*$-representation, respecting the filtered algebra structure. Denote the filtration by*

$$\ldots \subset F_{i-1} \subset F_i \subset \ldots$$

Suppose that $\mathbb{C}^*$ acts on F_i/F_{i-1} through $\lambda \mapsto \lambda^i$. Let $H = \bigoplus_j H_j$ be the grading induced by the $\mathbb{C}^*$-representation, where H_j is the eigenspace of the character $\lambda \mapsto \lambda^j$. Then for all i we have $F_i = \bigoplus_{j \leq i} H_j$, so that we have

$$H = \operatorname{gr} H \,,$$

as graded algebras. Here the grading on H comes from the $\mathbb{C}^*$-representation, and the grading on $\operatorname{gr} H$ from the filtration on H. $\square$

Example 1.7 Consider the $\mathbb{G}_\mathrm{m}$-action and equivariant vector field W on $\mathbb{P}^n$ of Example 1.3. Then, because the zero locus Z of V has dimension zero, the Koszul complex $K_W^\bullet$ is a resolution of $\mathcal{O}_Z$, and so we have that

$$
\begin{aligned}
\mathbb{H}^0(\mathbb{P}^n, K_W^\bullet) &= \Gamma(Z, \mathcal{O}_Z) \\
&= \mathbb{C}[s_1, \ldots, s_n]/(-s_1 s_n, s_2 - s_1^2, s_3 - s_1 s_2, \ldots, s_n - s_1 s_{n-1}) \\
&= \mathbb{C}[s_1]/(-s_1^{n+1}) \,.
\end{aligned}
$$

Note how the latter relations serve to recursively eliminate $s_2, \ldots, s_n$, leaving only the first generator s_1 and the first relation $-s_1 s_n$.

Remark 1.8 The assumption that $H^p(X, \Omega^q) = 0$, for all $p \neq q$, was made largely for simplicity, and because it is satisfied in all cases considered in this paper. It seems likely that the E_1-spectral sequence abutting to $\mathbb{H}(X, K_V^\bullet)$ always degenerates. (See [6], where this is proved for the case of a compact Kähler manifold X and a vector field V with non-empty zero set). If this is the case, then $\mathbb{H}^*(X, K_V^\bullet)$ is a doubly graded $\mathbb{C}$-algebra and isomorphic to $H_{DR}(X) = \bigoplus_{p,q} H^p(X, \Omega^q)$ as such.

1.2 Equivariant actions on vector bundles

To understand Chern classes in the context of Proposition 1.5, we need to study actions of vector fields on vector bundles. First we recall this concept without the presence of a $\mathbb{G}_\mathrm{m}$-action. Then we consider the $\mathbb{G}_\mathrm{m}$-equivariant case.

Actions of vector fields on vector bundles

Let X be a smooth Deligne-Mumford stack and E a vector bundle on X.

Definition 1.9 (Carrell-Lieberman [7]) Let V be a vector field on X. An *action* of V on E is a homomorphism of sheaves of $\mathbb{C}$-vector spaces

$$\widetilde{V} : E \longrightarrow E \,,$$

satisfying the Leibniz rule

$$\widetilde{V}(fe) = V(f)e + f\widetilde{V}(e) \,,$$

for all local sections f of $\mathcal{O}_X$ and e of E. Here we interpret V as a $\mathbb{C}$-linear derivation $V : \mathcal{O}_X \to \mathcal{O}_X$.

Note that any two actions of V on E differ by a homomorphism of vector bundles $E \to E$.

Remark 1.10 An action $\widetilde{V}$ of V on E is the same thing as a $\mathbb{G}_{\mathrm{m}}$-invariant vector field $\overline{V}$ on E lifting the vector field V on X. Here we mean $\mathbb{G}_{\mathrm{m}}$-invariant with respect to the natural fiber-wise action of $\mathbb{G}_{\mathrm{m}}$ on E by scalar multiplication. This means that in local (linear) coordinates on E, the coefficients of $\overline{V}$ are linear in these 'vertical' coordinates.

The action $\widetilde{V}$ is given in terms of the invariant lift $\overline{V}$ by

$$\widetilde{V}(e) = De(V) - e^*(\overline{V}) \,,$$

for any local section $e : X \to E$. This is an equality of sections of $e^* \mathcal{T}_E$. The bundle $e^* \mathcal{T}_E$ fits into the short exact sequence

$$0 \longrightarrow E \longrightarrow e^* \mathcal{T}_E \longrightarrow \mathcal{T}_X \longrightarrow 0 \,,$$

which is canonically split by $De : \mathcal{T}_X \to e^* \mathcal{T}_E$. We will often identify the invariant lift $\overline{V}$ and the action $\widetilde{V}$.

If the vector field V on X comes about by differentiating a $\mathbb{G}_{\mathrm{a}}$-action, and the vector field $\overline{V}$ on E comes about by differentiating a compatible *linear* $\mathbb{G}_{\mathrm{a}}$-action on E, then $\overline{V}$ is $\mathbb{G}_{\mathrm{m}}$-invariant, and therefore gives rise to an action of V on E.

Examples 1.11 1. Let $\nabla : E \to \Omega_X \otimes E$ be a connection on E. Then, via ∇, every vector field V acts on E. Just set $\widetilde{V}$ equal to the covariant derivative ∇_V.

2. Given the vector field V on X, the vector bundles $\mathcal{O}_X$, $\mathcal{T}_X$ and Ω_X have natural V-actions. For $\mathcal{O}_X$, we have $\widetilde{V} = V$, for $\mathcal{T}_X$, we have that $\widetilde{V}$ is equal to the Lie derivative with respect to V and for Ω_X we have $\widetilde{V} = d \circ V$.

3. Given actions of V on the vector bundles E and F, there are natural induced actions on $E \otimes F$ and $\mathcal{H}om(E, F)$. These are given by the usual Leibniz formulas

$$\widetilde{V}(e \otimes f) = \widetilde{V}(e) \otimes f + e \otimes \widetilde{V}(f)$$
$$\widetilde{V}(\phi)(e) = \widetilde{V}\big(\phi(e)\big) - \phi\big(\widetilde{V}(e)\big) \,.$$

4. Consider the vector field W on $\mathbb{P}^n$ from Example 1.3. It acts on $\mathcal{O}(1)$ by the formula

$$\widetilde{W}(x_i) = \begin{cases} x_{i+1} & \text{if } i < n \\ 0 & \text{if } i = n \,. \end{cases}$$

We get induced actions of W on $\mathcal{O}(m)$, for all $m \in \mathbb{Z}$.

Remark 1.12 Recall how connections on E can be described in terms of splittings of the short exact sequence of $\mathcal{O}_X$-modules (the Atiyah extension)

$$0 \longrightarrow \Omega_X \otimes E \longrightarrow At(E) \longrightarrow E \longrightarrow 0 \,, \tag{4}$$

where $At(E)$ is the $\mathcal{O}_X$-module whose underlying sheaf of $\mathbb{C}$-vector spaces is equal to $E \oplus (\Omega \otimes E)$ and whose $\mathcal{O}_X$-module structure is defined by

$$f * (e, \omega \otimes e') = (fe, f\omega \otimes e' + df \otimes e).$$

If we denote by s_0 the $\mathbb{C}$-linear splitting of (4) given by $e \mapsto (e, 0)$, then every $\mathcal{O}_X$-linear splitting s of (4) defines a connection on E by the formula $s = s_0 + \nabla$, and conversely, every $\mathcal{O}_X$-linear splitting of (4) comes from a unique connection on E in this way.

We can describe actions of V on E in a similar vein. We define a short exact sequence of $\mathcal{O}_X$-modules

$$0 \longrightarrow E \longrightarrow At_V(E) \longrightarrow E \longrightarrow 0, \tag{5}$$

where $At_V(E)$ is the $\mathcal{O}_X$-module whose underlying sheaf of $\mathbb{C}$-vector spaces is $E \oplus E$ and whose $\mathcal{O}_X$-modules structure is given by

$$f * (e, e') = (fe, fe' + V(f)e).$$

Moreover, the inclusion in (5) is given by $e' \mapsto (0, e')$ and the quotient map by $(e, e') \mapsto e$. Again, let us denote the $\mathbb{C}$-linear splitting $e \mapsto (e, 0)$ of (5) by s_0. Then actions of V on E and $\mathcal{O}_X$-linear splittings of (5) correspond bijectively to each other via the formula $s = s_0 + \widetilde{V}$.

Note that pushing out (4) via the $\mathcal{O}_X$-homomorphism $V \otimes \mathrm{id}_E : \Omega_X \otimes E \to E$ gives (5). In other words, we have a homomorphism of short exact sequences of $\mathcal{O}_X$-modules

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \Omega_X \otimes E & \longrightarrow & At(E) & \longrightarrow & E & \longrightarrow & 0 \\
& & \downarrow{\scriptstyle V \otimes \mathrm{id}} & & \downarrow & & \downarrow{\scriptstyle \mathrm{id}} & & \\
0 & \longrightarrow & E & \longrightarrow & At_V(E) & \longrightarrow & E & \longrightarrow & 0.
\end{array}
\tag{6}
$$

Thus every action $\widetilde{V}$ of V on E gives rise to an $\mathcal{O}_X$-linear map (the Carrell-Lieberman map)

$$CL(\widetilde{V}) : At(E) \longrightarrow E \tag{7}$$
$$(e, \omega \otimes e') \longmapsto \langle \omega, V \rangle e' - \widetilde{V}(e),$$

making the diagram

$$
\begin{array}{ccc}
\Omega_X \otimes E & \longrightarrow & At(E) \\
\downarrow{\scriptstyle V \otimes \mathrm{id}} & \swarrow{\scriptstyle CL(\widetilde{V})} & \\
E & &
\end{array}
$$

commute.

13

The equivariant case

Now suppose that X is endowed with a $\mathbb{G}_{\mathrm{m}}$-action, which has been lifted to an action of $\mathbb{G}_{\mathrm{m}}$ on E by linear isomorphisms. Given a vector field V on X, and an action $\widetilde{V}$ of V on E, then $^{\lambda}\widetilde{V}$ is an action of $^{\lambda}V$ on E. So if V is equivariant, it is natural to consider equivariant lifts $\widetilde{V}$, which satisfy $^{\lambda}\widetilde{V} = \lambda\widetilde{V}$.

Remark Let $\overline{V}$ be an invariant vector field on E (with respect to the linear action), lifting the vector field V on X. Let $\widetilde{V}$ be the associated action of V on E. Then $\overline{V}$ is equivariant as a vector field on E (with respect to the geometric action) if and only if $\widetilde{V}$ is equivariant as a $\mathbb{C}$-linear homomorphism from E to E.

Hence, if the $\mathbb{G}_{\mathrm{m}}$-action and the equivariant vector field V on X come from an action of the group G on X as in Remark 1.2, then any lift of the G-action to a vector bundle E over X gives rise to an equivariant action of V on E.

If $\widetilde{V}$ and V' are two equivariant actions of V on E, then the difference $h = \widetilde{V} - V'$ is an equivariant homomorphism.

Example 1.13 If ∇ is an *invariant* connection on E, and V an equivariant vector field on X, then the covariant derivative ∇_V is an equivariant action of V on E.

Example 1.14 Suppose that E is trivial and that there exists a trivialization of E by a basis of global sections (e_i), which are *homogeneous*, i.e., we have

$$^{\lambda}e_i = \lambda^{r_i}e_i\,,$$

for all i and certain integers r_i, with respect to the geometric $\mathbb{G}_{\mathrm{m}}$-action. Then the connection on E induced by this trivialization via the formula $\nabla(e_i) = 0$ is invariant, and hence the covariant derivative with respect to an equivariant vector field is equivariant.

Example 1.15 Consider the $\mathbb{G}_{\mathrm{m}}$-action on $\mathbb{P}^n$ from Example 1.3. It lifts naturally to $\mathcal{O}(1)$ by the formula

$$^{\lambda}x_i = \lambda^i x_i\,.$$

The action $\widetilde{W}$ of W on $\mathcal{O}(1)$ given by $\widetilde{W}(x_i) = x_{i+1}$ $(x_{n+1} = 0)$ is equivariant with respect to this $\mathbb{G}_{\mathrm{m}}$-action. The same is true for the induced actions on $\mathcal{O}(m)$, for all $m \in \mathbb{Z}$.

Remark 1.16 Considering the natural induced action of $\mathbb{G}_{\mathrm{m}}$ on the short exact sequence (4), we note that all maps in (4) are $\mathbb{G}_{\mathrm{m}}$-invariant, and that invariant splittings correspond to invariant connections.

Denote by $E^{(-1)}$ the vector bundle E with the geometric $\mathbb{G}_{\mathrm{m}}$-action modified by the linear $\mathbb{G}_{\mathrm{m}}$-action in such a way that $^{\lambda}(e^{(-1)}) = \lambda^{\lambda}e$, where we have

denoted by $e^{(-1)}$ the section e of E considered as a section of $E^{(-1)}$. We introduce the $\mathbb{G}_{\mathrm{m}}$-action on (5) indicated by

$$0 \longrightarrow E \longrightarrow E^{(-1)} \oplus E \longrightarrow E^{(-1)} \longrightarrow 0 \,.$$

This choice of $\mathbb{G}_{\mathrm{m}}$-action is necessary to ensure that the formula

$$^\lambda\big(f * (e, e')\big) = (^\lambda f) * {}^\lambda(e, e')$$

holds, for the $\mathcal{O}_X$-action on $At_V(E)$.

Now invariant splittings of (5) correspond to equivariant actions of V on E. Moreover, the homomorphism of short exact sequences (6) is of degree one (or equivariant, in our language). The homomorphism $CL(\widetilde{V}) : At(E) \to E$ given by an equivariant action of V on E is equivariant: $^\lambda\big(CL(\widetilde{V})\big) = \lambda CL(\widetilde{V})$.

For future reference, we need some facts about the functorial behavior of equivariant actions on vector bundles.

Lemma 1.17 (pullbacks) *Let $f : X \to Y$ be a morphism of smooth Deligne-Mumford stacks. Let E be a vector bundle over Y. Assume that $\mathbb{G}_{\mathrm{m}}$ acts compatibly on X, Y and E. Let V be an equivariant vector field on X and W an equivariant vector field on Y, such that $Df(V) = W$. Let $\widetilde{W}$ be an equivariant action of W on E. Then there is an induced equivariant action $\widetilde{V}$ of V on f^*E, such that*

$$\widetilde{V}\big(f^*(e)\big) = f^*\big(\widetilde{W}(e)\big) \,,$$

for every local section e of E. If V, W and $\widetilde{W}$ come from compatible actions of G on X, Y and E, then $\widetilde{V}$ comes from the induced G-action on f^*E.

Lemma 1.18 (pushforward) *Let $\pi : X \to Y$ be a flat and proper morphism of smooth Deligne-Mumford stacks. Let E be a vector bundle on X, such that π_*E is a vector bundle on Y. Assume that $\mathbb{G}_{\mathrm{m}}$ acts compatibly on X, Y and E. Let V be an equivariant vector field on X and W an equivariant vector field on Y, such that $D\pi(V) = W$. Let $\widetilde{V}$ be an equivariant action of V on E. Then there is an induced equivariant action $\widetilde{W}$ of W on π_*E, such that*

$$\widetilde{W}(e) = \widetilde{V}(e) \,,$$

for every local section e of π_*E. If V, W and $\widetilde{V}$ come from compatible actions of G on X, Y and E, then $\widetilde{W}$ comes from the induced G-action on π_*E.

1.3 Chern classes

Now suppose that X is a smooth Deligne-Mumford stack and E a vector bundle on X of rank r. Suppose given compatible $\mathbb{G}_{\mathrm{m}}$-actions on X and E. Finally, let V be an equivariant vector field on X.

We shall now tensor the Koszul complex $K_V^\bullet$ over $\mathcal{O}_X$ with the sheaf of $\mathcal{O}_X$-modules $\mathcal{H}om(E, E)$. Thus $K_V^\bullet \otimes \mathcal{H}om(E, E)$ is a sheaf of differential graded

modules over the sheaf of differential graded $\mathcal{O}_X$-algebras $K_V^\bullet$. Hence the hypercohomology $\mathbb{H}^0\big(X, K_V^\bullet \otimes \mathcal{H}om(E,E)\big)$ is a module over the $\mathbb{C}$-algebra $\mathbb{H}^0(X, K_V^\bullet)$.

A $\mathbb{C}^*$-representation on $\mathbb{H}^0\big(X, K_V^\bullet \otimes \mathcal{H}om(E,E)\big)$ is given by the composition

$$\mathbb{H}^0\big(X, K_V^\bullet \otimes \mathcal{H}om(E,E)\big) \xrightarrow{\lambda^*} \mathbb{H}^0\big(X, \lambda^* K_V^\bullet \otimes \lambda^* \mathcal{H}om(E,E)\big)$$
$$\xrightarrow{\psi_\lambda \otimes \rho} \mathbb{H}^0\big(X, K_V^\bullet \otimes \mathcal{H}om(E,E)\big),$$

where ψ_λ is the homomorphism given by (3) and $\rho : \lambda^* \mathcal{H}om(E,E) \to \mathcal{H}om(E,E)$ is the natural isomorphism induced by the isomorphism $\lambda^* E \cong E$ given by the (geometric) action of $\mathbb{G}_\mathrm{m}$ on E.

Note that the $\mathbb{C}^*$ action on $\mathbb{H}^0\big(X, K_V^\bullet \otimes \mathcal{H}om(E,E)\big)$ is compatible with the $\mathbb{H}^0(X, K_V^\bullet)$-action. Thus, via this $\mathbb{C}^*$-representation, $\mathbb{H}^0\big(X, K_V^\bullet \otimes \mathcal{H}om(E,E)\big)$ becomes a graded module over the graded $\mathbb{C}$-algebra $\mathbb{H}^0(X, K_V^\bullet)$.

Now let $\widetilde{V}$ be an equivariant action of V on E. The Carrell-Lieberman homomorphism $CL(\widetilde{V})$ of (7) gives rise to a homomorphism of complexes

which we can view as a homomorphism in the derived category from E to $K_V^\bullet \otimes E$, because $[\Omega_X \otimes E \to At(E)]$ is a resolution of E. We denote this homomorphism by

$$c_{\widetilde{V}}(E) \in \mathrm{Hom}_{D(\mathcal{O}_X)}(E, K_V^\bullet \otimes E).$$

Via the canonical identification

$$\mathrm{Hom}_{D(\mathcal{O}_X)}(E, K_V^\bullet \otimes E) = \mathbb{H}^0\big(X, K_V^\bullet \otimes \mathcal{H}om(E,E)\big),$$

$c_{\widetilde{V}}(E)$ gives rise to a hypercohomology class, which we shall also denote by

$$c_{\widetilde{V}}(E) \in \mathbb{H}^0\big(X, K_V^\bullet \otimes \mathcal{H}om(E,E)\big)$$

and call the *Carrell-Lieberman class*.

It follows directly from Remark 1.16 that the Carrell-Lieberman class $c_{\widetilde{V}}(E)$ is a degree one element of $\mathbb{H}^0\big(X, K_V^\bullet \otimes \mathcal{H}om(E,E)\big)$.

Now assume given a regular function $Q : M(r \times r) \to \mathbb{A}^1$ of degree p, which is invariant under conjugation. This function gives rise to a morphism of X-schemes

$$Q_E : \mathcal{H}om(E,E) \longrightarrow \mathcal{O}_X,$$

by associating to an endomorphism ϕ of E the number we get by applying Q to any matrix representation of ϕ. The X-morphism Q_E corresponds to

a morphism of $\mathcal{O}_X$-algebras $\mathcal{O}_X[t] \to \mathrm{Sym}\,\mathcal{H}om(E,E)^\vee$, where t is a coordinate on $\mathbb{A}^1$. Evaluating at t, this morphism gives rise to a global section of $\mathrm{Sym}^p \mathcal{H}om(E,E)^\vee$, or equivalently, a symmetric p-linear homomorphism

$$Q' : \mathrm{Sym}^p \mathcal{H}om(E,E) \longrightarrow \mathcal{O}_X\,.$$

If ϕ is a local section of $\mathcal{H}om(E,E)$, then the regular function $Q_E \circ \phi$ on X is equal to $Q'(\phi^p)$.

Recall how characteristic classes are defined in terms of the Atiyah class. The Atiyah class $c(E) \in H^1\big(X, \Omega \otimes \mathcal{H}om(E,E)\big)$ is the cohomology class corresponding to the extension of $\mathcal{O}_X$-modules $At(E)$ given by (4) under the identification

$$\mathrm{Ext}^1(E, \Omega_X \otimes E) = H^1\big(X, \Omega_X \otimes \mathcal{H}om(E,E)\big)\,.$$

It gives rise to an element

$$c(E)^{\cup p} \in H^p\big(X, \Omega^{\otimes p} \otimes \mathcal{H}om(E,E)^{\otimes p}\big)\,,$$

by taking cup products. Then we apply the map induced on H^p by the composition

$$\Omega^{\otimes p} \otimes \mathcal{H}om(E,E)^{\otimes p} \longrightarrow \Omega^p \otimes \mathrm{Sym}^p \mathcal{H}om(E,E) \overset{\mathrm{id}\,\otimes Q'}{\longrightarrow} \Omega^p \otimes \mathcal{O}_X = \Omega^p$$

to $c(E)^{\cup p}$. We obtain $c^Q(E) \in H^p(X, \Omega^p)$, the characteristic class of E defined by Q:

$$c^Q(E) = H^p(\mathrm{id}\,\otimes Q')\big(c(E)^{\cup p}\big)\,.$$

For example, if Q is $(-1)^p$ times the degree $r-p$ coefficient of the characteristic polynomial, then $c^Q(E)$ is the p-th Chern class of E.

Remark If X is a scheme, then under our identification of $H^p(X, \Omega^p)$ with the singular cohomology $H^p(X, \mathbb{C})$, these Chern classes correspond to the usual Chern classes.

Let us now apply a corresponding process to the Carrell-Lieberman class $c_{\widetilde{V}}(E)$. We start by taking the cup product of this class with itself p times:

$$c_{\widetilde{V}}(E)^{\cup p} \in \mathbb{H}^0\big(X, (K_V^\bullet)^{\otimes p} \otimes \mathcal{H}om(E,E)^{\otimes p}\big)\,.$$

Then we compose with the map induced on $\mathbb{H}^0$ by

$$(K_V^\bullet)^{\otimes p} \otimes \mathcal{H}om(E,E)^{\otimes p} \overset{\mu \otimes Q'}{\longrightarrow} K_V^\bullet \otimes \mathcal{O}_X = K_V^\bullet\,,$$

where $\mu : (K_V^\bullet)^{\otimes p} \to K_V^\bullet$ is the multiplication map induced from the algebra structure on $K_V^\bullet$. Thus we get the associated characteristic class

$$c_{\widetilde{V}}^Q(E) = \mathbb{H}^0(\mu \otimes Q')\big(c_{\widetilde{V}}(E)^{\cup p}\big) \in \mathbb{H}^0(X, K_V^\bullet)\,. \tag{8}$$

Proposition 1.19 *The class $c_{\widetilde{V}}^{Q}(E) \in \mathbb{H}^0(X, K_V^{\bullet})$ is homogeneous of degree degree p. If X is proper and satisfies $h^{ij}(X) = 0$, for all $i \neq j$, then under the identification of Proposition 1.5 we have*

$$c_{\widetilde{V}}^{Q}(E) = c^{Q}(E).$$

PROOF. Let us denote by $K_V^{\geqslant -1}$ the naïve cutoff of $K_V^{\bullet}$ at minus one. So $K_V^{\geqslant -1}$ denotes the two term complex $[\Omega_X \to \mathcal{O}_X]$ given by the vector field V. There is a canonical injection of complexes $K_V^{\geqslant -1} \to K_V^{\bullet}$, which also gives rise to an injection $K_V^{\geqslant -1} \otimes E \to K_V^{\bullet} \otimes E$. We also have a canonical projection $K_V^{\geqslant -1} \to \Omega_X[1]$, giving rise to $K_V^{\geqslant -1} \otimes E \to \Omega_X \otimes E[1]$. In the derived category, we get an induced diagram

$$\operatorname{Hom}_{D(\mathcal{O}_X)}(E, K_V^{\geqslant -1} \otimes E) \longrightarrow \operatorname{Hom}_{D(\mathcal{O}_X)}(E, K_V^{\bullet} \otimes E)$$
$$\downarrow$$
$$\operatorname{Hom}_{D(\mathcal{O}_X)}(E, \Omega_X \otimes E[1])$$

which we identify with the diagram

$$\operatorname{Hom}_{D(\mathcal{O}_X)}(E, K_V^{\geqslant -1} \otimes E) \longrightarrow \mathbb{H}^0\big(X, K_V^{\bullet} \otimes \mathcal{H}om(E, E)\big)$$
$$\downarrow$$
$$H^1\big(X, \Omega_X \otimes \mathcal{H}om(E, E)\big)$$

Directly from the construction, it follows that the Carrell-Lieberman class $c_{\widetilde{V}}(E)$ lifts to $\operatorname{Hom}_{D(\mathcal{O}_X)}(E, K_V^{\geqslant -1} \otimes E)$, and this lift maps to the Atiyah class:

$$CL(\widetilde{V}) \longmapsto c_{\widetilde{V}}(E)$$
$$\uparrow$$
$$c(E)$$

The claim follows. $\square$

1.4 Localization to the big cell

Let X be, as above, a smooth Deligne-Mumford stack with a $\mathbb{G}_{\mathrm{m}}$-action, and let E be a vector bundle to which the $\mathbb{G}_{\mathrm{m}}$-action has been lifted. Let V be an equivariant vector field on X and $\widetilde{V}$ an equivariant action of V on E. Finally, let us denote by $Z \subset X$ the closed substack defined by the vanishing of V: the structure sheaf $\mathcal{O}_Z$ of Z is defined to be the cokernel of the vector field $V: \Omega_X \to \mathcal{O}_X$. Another way to think of $\mathcal{O}_Z$ is as the zero-degree cohomology sheaf of $K_V^{\bullet}$.

There is a canonical morphism of sheaves of differential graded algebras $K_V^\bullet \to \mathcal{O}_Z$, inducing a canonical morphism of graded algebras

$$\mathbb{H}^0(X, K_V^\bullet) \longrightarrow \Gamma(Z, \mathcal{O}_Z).$$

Remark When we restrict to Z, the action $\widetilde{V}$ induces an $\mathcal{O}_Z$-linear homomorphism $\widetilde{V}_Z : E_Z \to E_Z$. Applying the invariant polynomial Q, we get a regular function

$$Q(\widetilde{V}_Z) \in \Gamma(Z, \mathcal{O}_Z).$$

Under the canonical morphism $\mathbb{H}^0(X, K_V^\bullet) \to \Gamma(Z, \mathcal{O}_Z)$ the characteristic class $c_{\widetilde{V}}^Q(E)$ maps to $Q(\widetilde{V}_Z)$.

If X is affine and E is trivial over X, trivialized by a homogeneous basis (e_i) as in Example 1.14, then we form the matrix M of $\widetilde{V}$ with respect to this basis. In other words, $M = (m_{ij})$ is a square matrix with entries $m_{ij} \in \Gamma(X, \mathcal{O}_X)$, characterized by

$$\widetilde{V}(e_i) = \sum_j m_{ji} e_j.$$

In this case we have

$$\Gamma(Z, \mathcal{O}_Z) = \Gamma(X, \mathcal{O}_X)/V\big(\Gamma(X, \Omega_X)\big)$$

and we can compute $Q(\widetilde{V}_Z) \in \Gamma(Z, \mathcal{O}_Z)$ simply as the congruence class of $Q(M) \in \Gamma(X, \mathcal{O}_X)$.

Note that if e_i is homogeneous of degree d_i, then m_{ij} is homogeneous of degree $d_i - d_j + 1$.

Example Returning to Example 1.7, using the equivariant action $\widetilde{W}$ of W on $\mathcal{O}(m)$ of Example 1.15, we see that a basis for $\mathcal{O}(m)$ over $\{x_0 = 1\} \subset \mathbb{P}^n$ is given by x_0^m. We have $\widetilde{W}(x_0^m) = m\frac{x_1}{x_0}x_0^m = ms_1 x_0^m$ in the notation of Example 1.7. Thus the matrix of $\widetilde{W}$ with respect to this basis is ms_1 and so the first Chern class $c_1\big(\mathcal{O}(m)\big) \in \mathbb{H}^0(\mathbb{P}^n, K_W^\bullet)$ is equal to ms_1.

We shall apply these ideas in the following context. The stack X will be proper, smooth and satisfy our assumption on the vanishing of off-diagonal Hodge numbers. The stack X will be endowed with a $\mathbb{G}_m$-action and several equivariant vector bundles E_i. Moreover, X will have an equivariant vector field V on it, with natural equivariant lifts $\widetilde{V}_i$ to the various E_i. We will construct an affine scheme T, with a lift of the $\mathbb{G}_m$-action to T and an étale morphism $T \to X$. Over T we trivialize all E_i using homogeneous bases. Finally, we choose a collection of invariant polynomials Q_i of various degrees, giving rise to characteristic classes $c_i = c_i^{Q_i}(E_i) \in H_{DR}(X)$.

We consider the natural morphism

$$H_{DR}(X) = \mathbb{H}^0(X, K_V^\bullet) \longrightarrow \mathbb{H}^0(T, K_V^\bullet) = \Gamma(T, \mathcal{O}_T)/V\big(\Gamma(T, \Omega_T)\big), \qquad (9)$$

and note that we can compute the images of c_i under (9) as $Q_i(M_i)$, where M_i is the matrix representation of $\widetilde{V}_i$ with respect to our homogeneous basis of $E_i|T$.

In the cases we consider, it will turn out that the map (9) is injective in sufficiently low degrees and that $H_{DR}(X)$ is generated (as a $\mathbb{C}$-algebra) by the classes c_i.

2   Preliminaries on stable maps

2.1   Stable maps to vector bundles

Let X be a smooth and proper algebraic variety and E a vector bundle over X, with structure morphism $p : E \to X$.

Note Pulling back via p induces an isomorphism of Picard groups $p^* : \mathrm{Pic}(X) \to \mathrm{Pic}(E)$ which preserves the ample line bundles. Hence we have a canonical isomorphism $H_2(E)^+ \to H_2(X)^+$ which we use to identify these two semi-groups.

Note Let $\widetilde{f} : C \to E$ be a morphism from a prestable marked curve (C, x) to E. Then $\widetilde{f}$ is stable of class β if and only if $p(\widetilde{f}) = p \circ \widetilde{f}$ is stable of class β. This is because $p(\widetilde{f})$ cannot contract any component of C which $\widetilde{f}$ does not already contract, because no component of C can map into a fiber of p, these fibers being affine.

Let

$$
\begin{array}{ccc}
\mathcal{C} & \xrightarrow{\ f\ } & X \\
{\scriptstyle \pi}\big\downarrow & & \\
\overline{M}_{g,n}(X, \beta) & &
\end{array}
$$

be the universal curve and universal stable map. We may view $\pi_* f^* E$ as a (relative) scheme over $\overline{M}_{g,n}(X, \beta)$. Then it represents the following functor:

$$
\pi_* f^* E(T) = \Gamma(\mathcal{C}_T, f^* E),
$$

for any $\overline{M}_{g,n}(X, \beta)$-scheme T. Here $\mathcal{C}_T$ abbreviates the pull-back of $\mathcal{C}$ to T. If $T \to \overline{M}_{g,n}(X, \beta)$ is given by (C, x, f), then

$$
\begin{aligned}
\pi_* f^* E(T) &= \Gamma(C, f^* E) \\
&= \{\widetilde{f} : C \to E \mid p(\widetilde{f}) = f\}.
\end{aligned}
$$

Hence $\pi_* f^* E$ as a stack in its own right (forgetting the $\overline{M}_{g,n}(X\beta)$-structure) associates to the $\mathbb{C}$-scheme T the groupoid of triples $(C, x, \widetilde{f})$, where (C, x) is a prestable marked curve and $\widetilde{f} : C \to E$ is a morphism such that $p(\widetilde{f}) : C \to X$

is a stable map of class β. by the above notes, this is equivalent to saying that $\widetilde{f}$ is a stable map of class β. Thus we conclude that

$$\pi_* f^* E = \overline{M}_{g,n}(E, \beta).$$

In particular, $\overline{M}_{g,n}(E, \beta)$ is an algebraic stack, representable over $\overline{M}_{g,n}(X, \beta)$, by an (abelian) cone (see [3] for this terminology). We note the following consequence:

Proposition 2.1 *If E is convex over X, i.e., $H^1(\mathbb{P}^1, f^* E) = 0$ for all morphisms $f : \mathbb{P}^1 \to X$, then $\overline{M}_{0,n}(E, \beta)$ is a vector bundle over $\overline{M}_{0,n}(X, \beta)$, canonically identified $\pi_* f^* E$. The rank of this vector bundle is $\langle c_1(E), \beta \rangle + \operatorname{rk} E$.*

Example 2.2 For $m \geq 0$, the vector bundle $E_m = \pi_* f^* \mathcal{O}_{\mathbb{P}^n}(m)$ over $\overline{M}_{0,\nu}(\mathbb{P}^n, d)$ has rank $md + 1$ and represents $\overline{M}_{0,\nu}(\mathcal{O}_{\mathbb{P}^n}(m), d)$.

2.2 Stable maps to $\mathbb{P}^\infty$

Let $n < m$ be integers and consider $\mathbb{P}^n$ as a subvariety of $\mathbb{P}^m$ via $\mathbb{P}^n = \{\langle x_0, \ldots, x_n, 0, \ldots, 0 \rangle\} \subset \mathbb{P}^m$. Let $H = \langle 0, \ldots, 0, x_{n+1}, \ldots, x_m \rangle\} \cong \mathbb{P}^{m-n-1}$. Let $U = \mathbb{P}^m - H$ and consider the projection with center H onto $\mathbb{P}^n$, which is defined on U and makes U a vector bundle over $\mathbb{P}^n$ of rank $m - n$, in fact this vector bundle is isomorphic to a direct sum of $m - n$ copies of $\mathcal{O}_{\mathbb{P}^n}(1)$.

By applying $\overline{M}_{0,\nu}(\,\cdot\,, d)$ to the diagram of varieties

$$
\begin{array}{ccc}
U & \hookrightarrow & \mathbb{P}^m \\
\downarrow & & \\
\mathbb{P}^n & &
\end{array}
$$

we get the diagram of stacks

$$
\begin{array}{ccc}
\overline{M}_{0,\nu}(U, d) & \overset{\iota}{\hookrightarrow} & \overline{M}_{0,\nu}(\mathbb{P}^m, d) \\
\rho \downarrow & & \\
\overline{M}_{0,\nu}(\mathbb{P}^n, d) & & .
\end{array}
$$

Here ρ is a vector bundle of rank $(d+1)(m-n)$. Let κ be the zero section of this vector bundle. The projection ρ is a homotopy equivalence implying that

$$\kappa^* : H^p\big(\overline{M}_{0,\nu}(U, d)_{\mathrm{an}}, \mathbb{C}\big) \overset{\sim}{\longrightarrow} H^p\big(\overline{M}_{0,\nu}(\mathbb{P}^n, d)_{\mathrm{an}}, \mathbb{C}\big)$$

is an isomorphism for all p.

The complement of $\overline{M}_{0,\nu}(U, d)$ in $\overline{M}_{0,\nu}(\mathbb{P}^m, d)$ consists of all stable maps to $\mathbb{P}^m$ whose image intersects H. The locus of these stable maps has codimension n in $\overline{M}_{0,\nu}(\mathbb{P}^m, d)$. Thus

$$\iota^* : H^p\big(\overline{M}_{0,\nu}(\mathbb{P}^m, d)_{\mathrm{an}}, \mathbb{C}\big) \longrightarrow H^p\big(\overline{M}_{0,\nu}(U, d)_{\mathrm{an}}, \mathbb{C}\big)$$

is an isomorphism for all $p \leq 2n - 2$, by cohomological purity.

We conclude that

$$(\iota\kappa)^* : H^p\big(\overline{M}_{0,\nu}(\mathbb{P}^m, d)_{\mathrm{an}}, \mathbb{C}\big) \longrightarrow H^p\big(\overline{M}_{0,\nu}(\mathbb{P}^n, d)_{\mathrm{an}}, \mathbb{C}\big) \qquad (10)$$

is an isomorphism for $p \leq 2n - 2$. This leads to the following definition:

Definition 2.3 For every $p \geq 0$ we define

$$H^p\big(\overline{M}_{0,\nu}(\mathbb{P}^\infty, d)_{\mathrm{an}}, \mathbb{C}\big) = H^p\big(\overline{M}_{0,\nu}(\mathbb{P}^n, d)_{\mathrm{an}}, \mathbb{C}\big)\,,$$

for any n such that $n \geq \frac{1}{2}(p+2)$.

It follows from the above considerations that any two different choices of n lead to canonically isomorphic definitions of $H^p\big(\overline{M}_{0,\nu}(\mathbb{P}^\infty, d)_{\mathrm{an}}, \mathbb{C}\big)$, for fixed p. Taking the direct sum over all p gives rise to the $\mathbb{C}$-algebra

$$H^*\big(\overline{M}_{0,\nu}(\mathbb{P}^\infty, d)_{\mathrm{an}}, \mathbb{C}\big)\,.$$

For any n we have the canonical restriction map

$$H^*\big(\overline{M}_{0,\nu}(\mathbb{P}^\infty, d)_{\mathrm{an}}, \mathbb{C}\big) \longrightarrow H^*\big(\overline{M}_{0,\nu}(\mathbb{P}^n, d)_{\mathrm{an}}, \mathbb{C}\big)$$

which is a $\mathbb{C}$-algebra morphism and an isomorphism in degrees less than n (recall our degree convention from Remark 1.1).

Remark 2.4 For every ν, d, n we have

$$H^*\big(\overline{M}_{0,\nu}(\mathbb{P}^n, d)_{\mathrm{an}}, \mathbb{C}\big) = H_{DR}\big(\overline{M}_{0,\nu}(\mathbb{P}^n, d)\big) = \bigoplus_p H^p\big(\overline{M}_{0,\nu}(\mathbb{P}^n, d), \Omega^p\big)\,.$$

The first equality follows from Remark 1.1. The second equality is proved using the technique of virtual Hodge polynomials to reduce to the strata of a suitable stratification. Use the stratification by topological type. The details are worked out by Getzler and Pandharipande in [9]. This justifies writing also

$$H^*\big(\overline{M}_{0,\nu}(\mathbb{P}^\infty, d)_{\mathrm{an}}, \mathbb{C}\big) = H_{DR}\big(\overline{M}_{0,\nu}(\mathbb{P}^\infty, d)\big) = \bigoplus_p H^p\big(\overline{M}_{0,\nu}(\mathbb{P}^\infty, d), \Omega^p\big)\,.$$

Remark 2.5 It is, in fact, true that

$$(\iota\kappa)^* : H^p\big(\overline{M}_{0,\nu}(\mathbb{P}^m, d)_{\mathrm{an}}, \mathbb{C}\big) \longrightarrow H^p\big(\overline{M}_{0,\nu}(\mathbb{P}^n, d)_{\mathrm{an}}, \mathbb{C}\big)$$

is *surjective* for all p. One way to prove this is as follows. Consider the $\mathbb{G}_{\mathrm{m}}$-action on $\overline{M}_{0,\nu}(\mathbb{P}^m, d)$ induced by the $\mathbb{G}_{\mathrm{m}}$-action on $\mathbb{P}^m$ given by

$$\langle x_0, \dots, x_n, x_{n+1}, \dots, x_m \rangle \cdot \lambda = \langle x_0, \dots, x_n, \lambda x_{n+1}, \dots, \lambda x_m \rangle\,.$$

The substack $\overline{M}_{0,\nu}(\mathbb{P}^n, d)$ is a fixed locus for this action and $\overline{M}_{0,\nu}(U, d)$ is the big Białynicki-Birula cell of this action. Using virtual Poincaré polynomials as in

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[9] one proves that all Białynicki-Birula cells have only even cohomology, which implies the claim. (All fixed loci can be described explicitly, showing that they are amenable to the techniques of [9]. See for example [12] or [2] for explicit descriptions of fixed loci of $\mathbb{G}_{\mathrm{m}}$-actions on stable map stacks.)

This result implies that

$$H^*\big(\overline{M}_{0,\nu}(\mathbb{P}^\infty,d)_{\mathrm{an}},\mathbb{C}\big) \longrightarrow H^*\big(\overline{M}_{0,\nu}(\mathbb{P}^n,d)_{\mathrm{an}},\mathbb{C}\big)$$

is an epimorphism of $\mathbb{C}$-algebras for all n.

As we do not have a reference for the Białynicki-Birula decomposition of a Deligne-Mumford stack, we are careful to point out where we use this result. The only places are, in fact, Corollary 4.18, Proposition 4.21, Corollary 4.26 and Proposition 4.27.

3 Parameterizing stable maps to $\mathbb{P}^n$

Let d be an integer greater than or equal to 3. We will prove that there is an open substack U of $\overline{M}_{0,0}(\mathbb{P}^n,d)$ which is a vector bundle over $[\overline{M}_{0,d}/S_d]$. In fact, $U = [T/S_d]$, where T is a vector bundle over the scheme $\overline{M}_{0,d}$. The whole stack $\overline{M}_{0,0}(\mathbb{P}^n,d)$ can be covered by substacks isomorphic to U.

One way to describe U is as follows: introduce on $\mathbb{P}^n$ a suitable $\mathbb{G}_{\mathrm{m}}$-action and consider the induced $\mathbb{G}_{\mathrm{m}}$-action on $\overline{M}_{0,0}(\mathbb{P}^n,d)$. Then consider to every fixed component U_0 of this $\mathbb{G}_{\mathrm{m}}$-action the associated substack U of all points that move to U_0 as $\lambda \to 0$, $\lambda \in \mathbb{G}_{\mathrm{m}}$. For a suitable component U_0, we have that U is open in $\overline{M}_{0,0}(\mathbb{P}^n,d)$ and that U_0 is isomorphic to $[\overline{M}_{0,d}/S_d]$.

The most important case is the case where $n = 1$. In this case U can simply be described as the stack of all stable maps unramified over $\infty \in \mathbb{P}^1$. The case of general n is easily reduced to this special case.

We start by defining a vector bundle T over $\overline{M}_{0,d}$ and constructing a stable map $(\widetilde{C}, f)$ to $\mathbb{P}^1$, parametrized by T.

3.1 For every i a degree 1 map to $\mathbb{P}^1$

Let T be a scheme and (C, x) a prestable curve of genus zero over T, with d marked points. We denote the structure map by $\pi : C \to T$ and x stands for the d-tuple of sections $x_i : T \to C$, $i = 1, \dots, d$. Consider the canonical line bundles

$$\omega_i = x_i^* \Omega_{C/T}$$

on T and their duals $L_i = \omega_i^\vee$. Let $D_i \hookrightarrow C$ be the effective Cartier divisor defined by the i-th section x_i. By $\mathcal{K}_C$ we denote the sheaf of total quotient rings of C and by $\mathcal{K}_C^*$ its sheaf of units.

Note that the sheaf $\pi_* \mathcal{O}(D_i)$ is locally free of rank 2, because the genus of C/T is 0.

We will now fix an index i and define a canonical homomorphism

$$h_i : L_i \longrightarrow \pi_* \mathcal{O}(D_i).$$

Note that h_i may equivalently be defined by a global section

$$h_i \in \Gamma(C, \pi^* \omega_i(D_i)).$$

Assume for the moment that there exists a global section $s \in \Gamma(C, \mathcal{K}_C^*)$, which in a neighborhood of D_i generates the ideal sheaf of D_i, and which does not vanish anywhere else, except at D_i. In other words, s is the reciprocal of a global section of $\pi_* \mathcal{O}(D_i)$ which is nowhere contained in the submodule $\pi_* \mathcal{O} = \mathcal{O}$. Then we define h_i by the formula

$$h_i := \left(\frac{1}{s} + \frac{1}{1-d} \sum_{j \neq i} \frac{1}{s(x_j)} \right) ds(x_i) . \tag{11}$$

Note that this makes sense, even if s has a pole at any of the x_j.

Lemma 3.1 *Equation (11) defines h_i independently of the choice of s.*

PROOF. Let t be another section of $\mathcal{K}_C^*$, satisfying the same conditions as s. Then $\frac{t}{s} \in \mathcal{K}^*$ is an element of $\mathcal{O}^*$ in a neighborhood of D_i and so we have

$$\begin{aligned} dt(x_i) &= (d\tfrac{t}{s}s)(x_i) \\ &= s(x_i)d\tfrac{t}{s}(x_i) + \tfrac{t}{s}(x_i)ds(x_i) \\ &= \frac{t}{s}(x_i)ds(x_i) . \end{aligned}$$

Note also that

$$\frac{1}{s} - \frac{1}{t}\frac{t}{s}(x_i)$$

is a *regular* function on C, since the only poles cancel out. Thus this regular function is constant on the fibers of π and so it is equal to its evaluation at any of the sections x_j. In other words,

$$\frac{1}{s} - \frac{1}{t}\frac{t}{s}(x_i) = \frac{1}{s(x_j)} - \frac{1}{t(x_j)}\frac{t}{s}(x_i) ,$$

which, again, also makes sense if s or t has a pole at x_j.

We may now calculate as follows:

$$\left(\frac{1}{t} + \frac{1}{1-d} \sum_{j \neq i} \frac{1}{t(x_j)} \right) dt(x_i)$$

$$= \left(\frac{1}{t} + \frac{1}{1-d} \sum_{j \neq i} \frac{1}{t(x_j)} \right) \frac{t}{s}(x_i)ds(x_i)$$

$$= \left(\frac{1}{s} - \left(\frac{1}{s} - \frac{1}{t}\frac{t}{s}(x_i) \right) + \frac{1}{1-d} \sum_{j \neq i} \frac{1}{t(x_j)}\frac{t}{s}(x_i) \right) ds(x_i)$$

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$$= \left(\frac{1}{s} + \frac{1}{1-d} \sum_{j\neq i} \left(\frac{1}{s} - \frac{1}{t}\frac{t}{s}(x_i) + \frac{1}{t(x_j)}\frac{t}{s}(x_i) \right) \right) ds(x_i)$$

$$= \left(\frac{1}{s} + \frac{1}{1-d} \sum_{j\neq i} \left(\frac{1}{s(x_j)} - \frac{1}{t(x_j)}\frac{t}{s}(x_i) + \frac{1}{t(x_j)}\frac{t}{s}(x_i) \right) \right) ds(x_i)$$

$$= \left(\frac{1}{s} + \frac{1}{1-d} \sum_{j\neq i} \frac{1}{s(x_j)} \right) ds(x_i) \,.$$

Thus, indeed, h_i is well-defined. $\square$

Corollary 3.2 *There exists a unique homomorphism*

$$h_i : L_i \longrightarrow \pi_* \mathcal{O}(D_i) \,,$$

such that the restriction of h_i to any open subset of T which admits an s as above is given by Formula (11). The inverse image under h_i of the submodule $\pi_* \mathcal{O} \subset \pi_* \mathcal{O}(D_i)$ is 0.

PROOF. Since Zariski-locally in T we can find an s as required, these locally defined h_i glue. $\square$

The basic properties of h_i are summarized in the following

Proposition 3.3 *Evaluating h_i at x_j, for $j \neq i$ defines canonical sections $h_i(x_j) \in \Gamma(T, \omega_i)$. We have*

$$\sum_{j\neq i} h_i(x_j) = 0 \,.$$

Evaluating h_i at x_i gives a canonical section of $\Gamma(T, \omega_i \otimes x_i^* \mathcal{O}(D_i)) = \Gamma(T, \mathcal{O}_T)$. We have

$$h_i(x_i) = 1 \,,$$

under this identification.
Finally, h_i is characterized completely by these two properties.

PROOF. The two properties mentioned follow directly from the explicit definition of h_i in terms of a local parameter given above. The fact that h_i is determined by two properties follows from the fact that $\pi_* \mathcal{O}(D_i)$ is of rank two. $\square$

Let us now suppose given a section $\tau \in \Gamma(T, L_i)$. Then $h_i(\tau) \in \Gamma(T, \pi_* \mathcal{O}(D_i)) = \Gamma(C, \mathcal{O}(D_i))$ is a meromorphic function on C, which we may view as a rational map

$$h_i(\tau) : C \dashrightarrow \mathbb{P}^1 \quad .$$

This rational map is defined by the pencil given by the global sections $1, h_i(\tau)$ of $\mathcal{O}(D_i)$ (at least in the case where these global sections define a pencil). Note

that if τ is nowhere vanishing, then $h_i(\tau) : C \to \mathbb{P}^1$ is an everywhere defined morphism.

For simplicity, assume now that T is smooth and that $V = \{\tau = 0\}$ is a Cartier divisor. Then $V_C = \pi^{-1}V$ is also a Cartier divisor. Consider the closed subscheme $Z_i = D_i \cap V_C$ of C. Étale locally in C, we can find coordinates $(t_1, \dots, t_r)$ for T and $(t_1, \dots, t_r, s)$ for C such that D_i is given by $s = 0$ and V_C by $t_1 = 0$.

Thus the structure of the blow-up $\widetilde{C}$ of C along $Z_i = \{s = t_1 = 0\}$ is transparent: $\widetilde{\pi} : \widetilde{C} \to T$ is again a family of genus 0 prestable curves. The divisors D_j, for $j \neq i$ are contained in the locus where $\widetilde{C} \to C$ is an isomorphism. Thus they are divisors on $\widetilde{C}$ and are images of sections $x_j : T \to \widetilde{C}$. The strict transform $\widetilde{D}_i$ of D_i is the image of another section $\widetilde{x}_i : T \to \widetilde{C}$. The pair $(\widetilde{C}, \widetilde{x})$, where $\widetilde{x}_j = x_j$, for $j \neq i$, is a prestable marked curve. Moreover, the blow up morphism $p : \widetilde{C} \to C$ is a morphism of prestable curves, in particular, $p(\widetilde{x}_i) = x_i$.

Let $\widetilde{L}_i = \widetilde{\omega}_i^\vee$, where $\widetilde{\omega}_i = \widetilde{x}_i^* \Omega_{\widetilde{C}/T}$. There is a natural exact sequence

$$0 \longrightarrow \widetilde{L}_i \longrightarrow L_i \longrightarrow \mathcal{O}_V \longrightarrow 0 \ ,$$

coming via $\widetilde{x}_i^*$ from the exact sequence

$$0 \longrightarrow \omega_{\widetilde{C}}^\vee \longrightarrow p^* \omega_C \longrightarrow \mathcal{O}_E \longrightarrow 0 \ ,$$

where $E \subset \widetilde{C}$ is the exceptional divisor. Thus $\widetilde{L}_i = L_i(-V)$. The section τ of L_i factors through the subsheaf $\widetilde{L}_i$; let us denote this section of $\widetilde{L}_i$ by $\widetilde{\tau}$. The section $\widetilde{\tau}$ is nowhere vanishing, so it trivializes $\widetilde{L}_i$.

The marked prestable curve $(\widetilde{C}, \widetilde{x})$ has the canonical morphism

$$\widetilde{h}_i : \widetilde{L}_i \longrightarrow \widetilde{\pi}_* \mathcal{O}(\widetilde{D}_i)$$

associated to it. Since $\widetilde{\tau}$ is nowhere vanishing, the associated meromorphic function $\widetilde{h}_i(\widetilde{\tau})$ defines a morphism

$$\widetilde{h}_i(\widetilde{\tau}) : \widetilde{C} \longrightarrow \mathbb{P}^1.$$

The base locus of the pencil defined by the global sections $1, h_i(\tau)$ of $\mathcal{O}_C(D_i)$ is equal to $Z_i = D_i \cap V_C$. The rational map $h_i(\tau) : C \dashrightarrow \mathbb{P}^1$ sends the divisor D_i to $\infty \in \mathbb{P}^1$ and the divisor V_C to $0 \in \mathbb{P}^1$.

We have a commutative diagram

$$
\begin{array}{ccc}
\widetilde{L}_i & \longrightarrow & L_i \\
{\scriptstyle \widetilde{h}_i} \downarrow & & \downarrow {\scriptstyle h_i} \\
\widetilde{\pi}_* \mathcal{O}(\widetilde{D}_i) & & \pi_* \mathcal{O}(D_i) \\
& \searrow \quad \swarrow & \\
& \pi_* \mathcal{K}_C &
\end{array}
$$

Thus the morphism $\widetilde{h}_i(\widetilde{\tau}) : \widetilde{C} \to \mathbb{P}^1$ is the morphism defined by the rational map $h_i(\tau) : C \dashrightarrow \mathbb{P}^1$ via blowing up the locus where it is undefined.

To abbreviate notation, let us write f for the morphism $\widetilde{h}_i(\widetilde{\tau}) : \widetilde{C} \to \mathbb{P}^1$.

Before we can state the next proposition about f, we need to recall a few facts about ramification. Let $\pi : C \to S$ be a family of prestable curves over a scheme S and let $f : C \to \mathbb{P}^1$ be a morphism. Then there is a canonical homomorphism of line bundles on C

$$f^*\Omega_{\mathbb{P}^1} \longrightarrow \omega_{C/S} \ ,$$

where $\omega_{C/S}$ is the relative dualizing sheaf of C over S (which is also equal to the determinant of the relative cotangent complex, which shows the existence of the homomorphism). The ideal sheaf

$$I = (f^*\Omega_{\mathbb{P}^1} : \omega_{C/S}) = \{a \in \mathcal{O}_C \mid a \cdot \omega_{C/S} \subset f^*\Omega_{\mathbb{P}^1}\}$$

defines a closed subscheme $R \subset C$ called the *ramification scheme* of f. There is a canonical exact sequence

$$f^*\Omega_{\mathbb{P}^1} \longrightarrow \omega_{C/S} \longrightarrow \omega_{C/S} \otimes \mathcal{O}_R \longrightarrow 0 \ . \qquad (12)$$

We say that f is *unramified* over $a \in \mathbb{P}^1$, if $f^{-1}(a) \cap R = \varnothing$. Note that this is an open condition in S. Moreover, f is unramified over $a \in \mathbb{P}^1$, if and only if $(f^*\Omega_{\mathbb{P}^1} \to \omega_{C/S})|f^{-1}(a)$ is surjective. This means that for every geometric point s of S, the map $f_s : C_s \to \mathbb{P}^1$ is unramified over a. In particular, $f_s^{-1}(a)$ consists of $\deg f_s$ distinct points.

Lemma 3.4 *Let f be unramified over $a \in \mathbb{P}^1$. Then $f^{-1}(a)$ is finite étale of degree $\deg f$ over S. If $x : S \to C$ is a section such that $f(x) = a$, then the derivative*

$$Df : \mathcal{T}_{C/S} \longrightarrow f^*\mathcal{T}_{\mathbb{P}^1}$$

pulls back via x to an isomorphism

$$Df(x) : x^*\mathcal{T}_{C/S} \longrightarrow \mathcal{T}_{\mathbb{P}^1}(a) \otimes \mathcal{O}_S \ ,$$

where $\mathcal{T}_{\mathbb{P}^1}(a)$ denotes the tangent space of $\mathbb{P}^1$ at a.

PROOF. The structure sheaf $\mathcal{O}_{f^{-1}(a)}$ of the inverse image $f^{-1}(a)$ fits into the exact sequence

$$0 \longrightarrow \mathcal{O}_C \longrightarrow f^*\mathcal{O}(1) \longrightarrow \mathcal{O}_{f^{-1}(a)} \longrightarrow 0 \,,$$

obtained from an identification $\mathcal{O}(1) = \mathcal{O}(a)$. We get an induced exact sequence

$$0 \longrightarrow \mathcal{O}_S \longrightarrow \pi_*f^*\mathcal{O}(1) \longrightarrow \pi_*\mathcal{O}_{f^{-1}(a)} \longrightarrow 0 \,.$$

Since this sequence stays exact after arbitrary base change, it proves that $\pi_*\mathcal{O}_{f^{-1}(a)}$ is locally free of rank $\deg f$ (recall that $\pi_*f^*\mathcal{O}(1)$ is locally free of rank $\deg f + 1$). By considering the fibers of $C \to S$ we see that $f^{-1}(a)$ is quasi-finite over S. Since it is also proper, it is finite, thus flat. Then étale is equivalent to unramified, which can be checked fiber-wise. $\square$

Now we come back to $f : \widetilde{C} \to \mathbb{P}^1$ over T, defined above.

Proposition 3.5 *The morphism $f : \widetilde{C} \to \mathbb{P}^1$ is a family of degree 1 maps to $\mathbb{P}^1$, unramified over $\infty \in \mathbb{P}^1$. We have*

$$f^{-1}(\infty) = \widetilde{D}_i \ ,$$

and

$$Df(\widetilde{x}_i)(\widetilde{\tau}) = \frac{\partial}{\partial z}(\infty) \otimes 1 \ , \tag{13}$$

where z is the canonical coordinate at $\infty \in \mathbb{P}^1$ and $\frac{\partial}{\partial z}(\infty)$ is the evaluation of $\frac{\partial}{\partial z}$ at $z = 0$.

PROOF. This follows directly from the construction. Formula (13) is a direct calculation. $\square$

3.2 A degree d map to $\mathbb{P}^1$

Now let us suppose given a regular function $b \in \Gamma(T, \mathcal{O})$ and for every $i = 1, \ldots, d$ a section $\tau_i \in \Gamma(T, L_i)$. Then for each i we have $h_i(\tau_i) \in \Gamma(C, \mathcal{O}(D_i))$ and so for the sum we have

$$b + \sum_{i=1}^{d} h_i(\tau_i) \in \Gamma\big(C, \mathcal{O}(\textstyle\sum_i D_i)\big) \ ,$$

which we may also view as a rational map

$$b + \textstyle\sum_{i=1}^{d} h_i(\tau_i) : C \dashrightarrow \mathbb{P}^1 \ . \tag{14}$$

Note that this is an everywhere defined morphism, if all of the τ_i are nowhere vanishing.

Let $V_i = \{\tau_i = 0\}$ and, as above,

$$Z_i = D_i \cap V_{i,C} \ .$$

Let also $Z = Z_1 \cup \ldots \cup Z_d$ be the union of these pairwise disjoint closed subschemes.

Let us assume, as above, that T is smooth and that the V_i are Cartier divisors. Let $\widetilde{C}$ be the blow-up of C along Z. We get induced sections $\widetilde{x}_i : T \to \widetilde{C}$, making $(\widetilde{C}, \widetilde{x})$ a prestable marked curve. We also get induced nowhere vanishing sections $\widetilde{\tau}_i \in \Gamma(T, \widetilde{L}_i)$ and hence an everywhere defined morphism

$$f = b + \sum_{i=1}^{d} \widetilde{h}_i(\widetilde{\tau}_i) : \widetilde{C} \longrightarrow \mathbb{P}^1 \ .$$

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We may also write

$$f = b + \sum_{i=1}^{d} f_i \ , \tag{15}$$

where f_i is the morphism $\widetilde{C} \to \mathbb{P}^1$, defined by $\widetilde{h}_i(\widetilde{\tau}_i)$, as above.

Note that the base locus of the pencil defined by the sections 1 and $b + \sum_i h_i(\tau_i)$ of $\mathcal{O}_C(\sum_i D_i)$ is Z and so $f : \widetilde{C} \to \mathbb{P}^1$ is the morphism defined by blowing up the locus of indeterminacy of rational map (14).

Proposition 3.6 *Assume that (C, x) is a stable marked curve. Then $(\widetilde{C}, f)$ is a stable map of degree d. The canonical morphism $p : \widetilde{C} \to C$ identifies (C, x) as the stabilization of $(\widetilde{C}, \widetilde{x})$. The morphism $f : \widetilde{C} \to \mathbb{P}^1$ is unramified over $\infty \in \mathbb{P}^1$ and*

$$f^{-1}(\infty) = \sum_{i=1}^{d} \widetilde{D}_i \ . \tag{16}$$

Assume now that all τ_i are nowhere vanishing (which implies that $\widetilde{C} = C$) and that all fibers of C are irreducible. Then we have

$$Df(x_i)(\tau_i) = \frac{\partial}{\partial z}(\infty), \ for \ all \ i = 1, \dots, d \tag{17}$$

and if we let R be the ramification scheme of f, then $f|R$ avoids ∞, thus is a regular function and we have

$$b = \frac{1}{2d - 2} \operatorname{tr}_{R/T}(f|R) \ . \tag{18}$$

Finally, if $g : C \to \mathbb{P}^1$ is another family of morphisms of degree d unramified over ∞ with Properties (16), (17) and (18), then $g = f$.

PROOF. The stability condition can be checked on fibers of $\widetilde{\pi}$. The only potentially unstable components in such a fiber (over t) come from an exceptional divisor of the blow up. Such a component $E_{i,t}$ is isomorphic to $\mathbb{P}^1$ and intersects a unique $\widetilde{D}_i$ at a unique point which gets mapped to ∞. The component $E_{i,t}$ also intersects $\widetilde{V}_{i,C}$, the strict transform of $V_{i,C}$, in a unique different point, which the map f_i sends to 0. Thus f_i is of degree 1, when restricted to $E_{i,t}$. Since all the other maps f_j, for $j \neq i$, are constant on $E_{i,t}$, we see that $f|E_{i,t}$ is of degree 1, and hence that $E_{i,t}$ is a stable component of the map $(\widetilde{C}, f)$.

Since $p : (\widetilde{C}, \widetilde{x}) \to (C, x)$ is a morphism of prestable marked curves, and (C, x) is stable, p has to be the stabilization morphism. This follows from the universal property of stabilization and the fact that every morphism of stable marked curves is an isomorphism (see [4], page 27).

By Proposition 3.5 each of the f_i is unramified over ∞ and maps $\widetilde{D}_i$ to ∞. Moreover, $\widetilde{\tau}_i$ gets mapped to the canonical tangent vector at $\infty \in \mathbb{P}^1$. But f_j

for $j \neq i$ is holomorphic (i.e., nowhere equal to ∞) in a neighborhood of $\widetilde{D}_i$, and so adding it to f_i does not affect these properties of f_i at $\widetilde{D}_i$. Hence f is unramified over ∞ and $f^{-1}(\infty) = \sum_i \widetilde{D}_i$. Moreover, the derivative of f has the same behavior at x_i as the derivative of f_i, and so Formula (17) follows from Proposition 3.5.

Now we assume that all τ_i are nowhere vanishing. This assumption we make for $\widetilde{C} \to C$ to be an isomorphism. We also assume that all the fibers of C are irreducible. This has the nice consequence that, at least locally in T, we can find an affine coordinate s for C, such that $s(x_i)$ has no poles, for any i . More precisely, we can write C as a product $T \times \mathbb{P}^1$, such that the sections x_i become functions $x_i : T \to \mathbb{P}^1$, and we can arrange things in such a way that x_i avoids $\infty \in \mathbb{P}^1$, for all i. Then we let s be the affine coordinate for $\mathbb{P}^1 - \infty$. Now the $s(x_i)$ are regular functions on T, which we abbreviate by

$$ a_i = s(x_i) \ . $$

We may now use $s - a_i$ as equation for D_i, so that h_i (see (11)) becomes

$$ h_i = \left(\frac{1}{s - a_i} + \frac{1}{1-d} \sum_{j \neq i} \frac{1}{a_j - a_i} \right) ds(a_i) \ . $$

We may also use $ds(a_i)$ to trivialize ω_i and hence L_i. We write

$$ \tau_i = q_i \frac{\partial}{\partial s}(a_i) $$

in this trivialization. Thus

$$ h_i(\tau_i) = q_i \left(\frac{1}{s - a_i} + \frac{1}{1-d} \sum_{j \neq i} \frac{1}{a_j - a_i} \right) $$

and hence

$$ f(s) = b + \sum_i h_i(\tau_i) $$

$$ = b + \sum_i \frac{q_i}{s - a_i} + \frac{1}{1-d} \sum_{i \neq j} \frac{q_i}{a_j - a_i} \tag{19} $$

Suppose that $f(s)$ ramifies at $(r_\alpha)_{\alpha=1,\dots,2d-2}$. Then the (r_α) are the roots of $f'(s) = 0$. A quick calculation shows that

$$ f'(s) = - \sum_i \frac{q_i}{(s - a_i)^2} . $$

and hence $f'(s) = 0$ is equivalent to

$$ \sum_i q_i \prod_{j \neq i} (s - a_j)^2 = 0 \ . $$

Thus we see that

$$\sum_i q_i \prod_{j \neq i}(s - a_j)^2 = \left(\sum_i q_i\right) \prod_{\alpha=1}^{2d-2}(s - r_\alpha) , \qquad (20)$$

by comparing leading coefficients. By differentiating (20), we also get

$$2\sum_i q_i \sum_{j \neq i} \frac{1}{s - a_j} \prod_{k \neq i}(s - a_k)^2 = \left(\sum_i q_i\right) \sum_\alpha \prod_{\beta \neq \alpha}(s - r_\beta) . \qquad (21)$$

Now note that $\sum_\alpha \frac{1}{s-r_\alpha}$ is equal to the right hand side of (21) divided by the right hand side of (20), and so

$$\sum_\alpha \frac{1}{s - r_\alpha} = \frac{2 \sum_i q_i \prod_{k \neq i}(s - a_k)^2 \sum_{j \neq i} \frac{1}{s - a_j}}{\sum_i q_i \prod_{k \neq i}(s - a_k)^2} \qquad (22)$$

We are now ready to compute the trace of $f|R$.

$$\sum_\alpha f(r_\alpha) = \sum_\alpha \left(b + \sum_i \frac{q_i}{r_\alpha - a_i} + \frac{1}{1 - d} \sum_{i \neq j} \frac{q_i}{a_j - a_i} \right)$$

$$= (2d - 2)b - 2\sum_{i \neq j} \frac{q_i}{a_j - a_i} - \sum_i q_i \sum_\alpha \frac{1}{a_i - r_\alpha}$$

$$= (2d - 2)b - 2\sum_{i \neq j} \frac{q_i}{a_j - a_i}$$

$$\qquad - \sum_i q_i \frac{2 \sum_\ell q_\ell \prod_{k \neq \ell}(a_i - a_k)^2 \sum_{j \neq \ell} \frac{1}{a_i - a_j}}{\sum_\ell q_\ell \prod_{k \neq \ell}(a_i - a_k)^2}$$

(by Equation (22))

$$= (2d - 2)b - 2\sum_{i \neq j} \frac{q_i}{a_j - a_i}$$

$$\qquad - \sum_i q_i \frac{2q_i \prod_{k \neq i}(a_i - a_k)^2 \sum_{j \neq i} \frac{1}{a_i - a_j}}{q_i \prod_{k \neq i}(a_i - a_k)^2}$$

(because for $\ell \neq i$ the product vanishes)

$$= (2d - 2)b - 2\sum_{i \neq j} \frac{q_i}{a_j - a_i} - 2\sum_i q_i \sum_{j \neq i} \frac{1}{a_i - a_j}$$

$$= (2d - 2)b .$$

It remains to prove the uniqueness claim. But it is easy to see that

$$f(s) = c + \sum_{i=1}^{d} \frac{q_i}{s - a_i}$$

is the most general degree d morphism which maps a_i to ∞, for all i, and satisfies

$$\frac{d}{ds} \frac{1}{f(s)}\bigg|_{s=a_i} = \frac{1}{q_i} \ ,$$

for all i. Thus the uniqueness follows. $\square$

3.3   The universal situation

Consider $\overline{M}_{0,d}$, the scheme of stable curves of genus zero marked by $\underline{d} = \{1, \dots, d\}$. Let $\pi : C \to \overline{M}_{0,d}$ be the universal curve and $x_1, \dots, x_d$ the universal sections. Let ω_i and L_i be line bundles over $\overline{M}_{0,d}$ defined as above. Finally, define

$$T = \mathbb{A}^1 \times \prod_{i=1}^{d} L_i \ ,$$

where the product is taken over $\overline{M}_{0,n}$. Thus T is a vector bundle of rank $d+1$ over $\overline{M}_{0,d}$. In particular, T is a smooth scheme of dimension $d-3+d+1 = 2d-2$. When we pull back any of the $\overline{M}_{0,d}$-schemes L_i or C to T, we endow them with a subscript T (which we also occasionally omit).

If S is a scheme, then we may think of S-valued points of T as $2d+2$-tuples

$$(C, x, b, \tau) = (C, x_1, \dots, x_d, b, \tau_1, \dots, \tau_d) \ ,$$

where $(C, x) \in \overline{M}_{0,d}(S)$ is a stable marked curve over S, $b \in \mathbb{A}^1_S$ is a regular function on S and τ_i is a tangent vector of C at x_i, or rather a section of L_i over S. Similarly, S-valued points of C_T are $2d+3$-tuples

$$(C, x, b, \tau, \Delta),$$

where (C, x, b, τ) is as above and $\Delta \in C(S)$.

The various projections of $\mathbb{A}^1 \times \prod_{i=1}^{d} L_i$ onto its components define a canonical regular function $b \in \Gamma(T, \mathcal{O})$ and canonical sections $\tau_i \in \Gamma(T, L_{i,T})$. These are, of course, the universal b, τ. As in Section 3.2, we let $Z = Z_1 \cup \ldots \cup Z_d \subset C_T$, where $Z_i = D_i \cap \{\tau_i = 0\}$, and we blow up C_T at Z to obtain $\widetilde{\pi} : \widetilde{C} \to T$.

As in Section 3.1, we get for every i a meromorphic function $f_i = h_i(\tau_i) \in \Gamma(C_T, \mathcal{O}(D_i))$, which we can identify with the meromorphic function $\widetilde{h}_i(\widetilde{\tau}_i) \in \Gamma(\widetilde{C}, \mathcal{O}(\widetilde{D}_i))$. Note that f_i is an everywhere non-vanishing section of the line bundle $\mathcal{O}(\widetilde{D}_i)$. We also get for every $i \neq j$ a regular function

$$\eta_{ij} = f_i(x_j) = h_i(x_j)(\tau_i) = \widetilde{h}_i(\widetilde{\tau}_i)(\widetilde{x}_j) \,. \tag{23}$$

The η_{ij} will be very useful in Section 4.

As in Section 3.2, we get a meromorphic function

$$b + \sum_{i=1}^{d} h_i(\tau_i) \in \Gamma\big(C_T, \mathcal{O}(\textstyle\sum_i D_i)\big) \ ,$$

and an induced morphism

$$f : \widetilde{C} \longrightarrow \mathbb{P}^1 \ .$$

By Proposition 3.6 $(\widetilde{C}, f)$ is a stable map of degree d over T, and so we get an induced morphism

$$T \longrightarrow \overline{M}_{0,0}(\mathbb{P}^1, d) \ . \tag{24}$$

The symmetric group S_d acts on $\overline{M}_{0,d}$ from the right by

$$(C, x_1, \ldots, x_d) \cdot \sigma = (C, x_{\sigma(1)}, \ldots, x_{\sigma(d)}) \ .$$

We have compatible actions on T given by

$$(C, x_1, \ldots, x_d, b, \tau_1, \ldots, \tau_d) \cdot \sigma = (C, x_{\sigma(1)}, \ldots, x_{\sigma(d)}, b, \tau_{\sigma(1)}, \ldots, \tau_{\sigma(d)})$$

and on C_T given by

$$(C, x_1, \ldots, x_d, b, \tau_1, \ldots, \tau_d, \Delta) \cdot \sigma = (C, x_{\sigma(1)}, \ldots, x_{\sigma(d)}, b, \tau_{\sigma(1)}, \ldots, \tau_{\sigma(d)}, \Delta) \ .$$

The vector bundle $\prod_{i=1}^{d} L_i$ over $\overline{M}_{0,d}$ descends to a vector bundle $E = [\prod L_i / S_d]$ of rank d over $[\overline{M}_{0,d}/S_d]$, which does not split anymore. But still, $[T/S_d] \to [\overline{M}_{0,d}/S_d]$ is a vector bundle of rank $d + 1$.

For all $\sigma \in S_d$, the induced automorphism of C_T identifies Z_i with $Z_{\sigma(i)}$ and so $Z \subset C_T$ is an invariant subscheme under the S_d-action. Thus we get an induced action of S_d on the blow-up $\widetilde{C}$. This action is compatible with the projection $\widetilde{\pi} : \widetilde{C} \to T$ and so we get an induced prestable curve $[\widetilde{C}/S_d] \to [T/S_d]$. Note also that $f : \widetilde{C} \to \mathbb{P}^1$ is S_d-invariant. This follows from Formula (15) and the fact that the action of σ exchanges f_i and $f_{\sigma(i)}$. Thus we get an induced morphism $[f/S_d] : [\widetilde{C}/S_d] \to \mathbb{P}^1$, and so we see that (24) induces a morphism

$$[T/S_d] \longrightarrow \overline{M}_{0,0}(\mathbb{P}^1, d) \ . \tag{25}$$

Theorem 3.7 *The morphism (25) is an isomorphism onto the open substack of $\overline{M}_{0,0}(\mathbb{P}^1, d)$ consisting of stable maps which are unramified over $\infty \in \mathbb{P}^1$.*

Before proving the theorem, we prepare a little more. We already remarked that 'unramified over ∞' is an open condition on stable maps to $\mathbb{P}^1$. So the stable maps unramified over ∞ form an open substack $U \subset \overline{M}_{0,0}(\mathbb{P}^1, d)$. Let z be the canonical coordinate at ∞ on $\mathbb{P}^1$. If $(C, f) \in \overline{M}_{0,0}(\mathbb{P}^1, d)(S)$ is a stable map parametrized by the scheme S, then $f^*(z)$ defines the closed subscheme $f^{-1}(\infty)$ of C.

Assume that the stable map (C, f) is unramified over ∞. Then $f^{-1}(\infty) \to S$ is finite étale of degree d. We call an isomorphism

$$\phi : \underline{d} \times S \longrightarrow f^{-1}(\infty)$$

an *indexing* of $f^{-1}(\infty)$. The local indexings form an S-scheme

$$P = \mathrm{Isom}_S(\underline{d} \times S, f^{-1}(\infty)) \ ,$$

which is a principal (right) S_d-bundle over S. In particular, $P \to S$ is finite étale of degree $d!$.

Let U' be the stack of triples (C, f, ϕ), where $(C, f) \in \overline{M}_{0,0}(\mathbb{P}^1, d)$ is a stable map unramified over ∞ and ϕ is an indexing of $f^{-1}(\infty)$. Thus $U' \to U$ is a principal S_d-bundle, in fact, $U' \to U$ is the stack of indexings of the universal $f^{-1}(\infty)$. It is not difficult to see that U' is a scheme.

Now consider the stable map $(\widetilde{C}, f)$ defined over T. By Proposition 3.6 we have that

$$f^{-1}(\infty) = \sum_{i=1}^{d} \widetilde{D}_i \ .$$

Thus $(\widetilde{C}, f)$ comes with a canonical indexing of $f^{-1}(\infty)$. Therefore, we get a morphism $T \to U'$. In other words, the morphism (24) lifts in a natural way to $U' \to \overline{M}_{0,0}(\mathbb{P}^1, d)$.

Theorem 3.7 now follows from the following proposition.

Proposition 3.8 *The canonical morphism* $T \to U'$ *is an isomorphism of schemes with* S_d*-action.*

PROOF. Let us define the inverse of $T \to U'$. Let (C, f, ϕ) be an S-valued point of U'. We need to associate to (C, f, ϕ) an S-valued point of T. Since U' is smooth, we may assume that S is smooth and that the structure morphism $S \to U'$ is étale.

Let $S' \subset S$ be the locus over which f does not contract any components of C. Since a contracted component has at least 3 special points, the complement of S' in S has codimension at least 3. Thus to define a regular function on S is equivalent to defining a regular function on S' (codimension 2 would suffice for this). Let $C' \to S'$ be the restriction of $C \to S$ to S'.

Let R be the ramification scheme of f. Over S' the exact sequence (12) is exact on the left also

$$0 \longrightarrow f^* \Omega_{\mathbb{P}^1} \longrightarrow \omega_{C'/S'} \longrightarrow \omega_{C'/S'} \otimes \mathcal{O}_{R'} \longrightarrow 0$$

and commutes with base change to the fibers of $C' \to S'$. Hence we see that over S' the pullback $R' = S' \times_S R$ is finite and flat of degree $2d - 2$ over S'.

By assumption, $R' \cap f^{-1}(\infty) = \varnothing$ and so $f|R'$ factors through $\mathbb{A}^1 = \mathbb{P}^1 - \infty$, i.e., $f|R'$ is a regular function on R'. Then we can take the trace to get a regular function

$$\mathrm{tr}_{R'/S'}(f|R')$$

on S', which extends uniquely to a regular function on S, which we denote by $\mathrm{tr}_{R/S}(f|R)$, by abuse of notation. Define

$$b = \frac{1}{2d-2} \, \mathrm{tr}_{R/S}(f|R) \ .$$

Thus b is the average of the ramification points of the stable map f.

The isomorphism $\phi : \underline{d} \times S \to f^{-1}(\infty)$ defines d sections $x_1, \dots, x_d : S \to C$. Consider the derivative

$$Df : \mathcal{T}_{C/S} \longrightarrow f^* \mathcal{T}_{\mathbb{P}^1}$$

and pull it back via x_i to get an isomorphism

$$Df(x_i) : L_i = x_i^* \mathcal{T}_{C/X} \longrightarrow x_i^* f^* \mathcal{T}_{\mathbb{P}^1} = \mathcal{T}_{\mathbb{P}^1}(\infty) \otimes \mathcal{O}_S \ ,$$

where $\mathcal{T}_{\mathbb{P}^1}(\infty)$ is the tangent space of $\mathbb{P}^1$ at ∞ (see Lemma 3.4). Taking the preimage of $\frac{\partial}{\partial z}(\infty) \otimes 1$ under $Df(x_i)$ yields a section $\tau_i \in \Gamma(S, L_i)$.

Stabilizing (C, x) defines a stable marked curve

$$(\overline{C}, \overline{x}) = (C, x)^{\mathrm{stab}} \ .$$

The stabilization morphism $p : C \to \overline{C}$ induces a homomorphism

$$L_i \to \overline{L}_i = \overline{x}_i^* \mathcal{T}_{\overline{C}/S} \ ,$$

which maps τ_i to a section $\overline{\tau}_i \in \Gamma(S, \overline{L}_i)$.

Now we have defined an S-valued point

$$(\overline{C}, \overline{x}_1, \dots, \overline{x}_d, b, \overline{\tau}_1, \dots, \overline{\tau}_d)$$

of T, which we declare to by the image of (C, f, ϕ), thus defining a morphism $U' \to T$.

Claim I. The composition $T \to U' \to T$ is equal to the identity.

To prove this claim, we start with an S-valued point (C, x, b, τ) of T. We pass to $(\widetilde{C}, \widetilde{x}, f)$, which defines a point of U'. From Proposition 3.6 it follows that (C, x) is the stabilization of $(\widetilde{C}, \widetilde{x})$, and so the marked curve associated to $(\widetilde{C}, \widetilde{x}, f)$ under $U' \to T$ is (C, x). It remains to prove that $b = \frac{1}{2d-2} \, \mathrm{tr}(f|R)$ and $\tau_i = Df(x_i)^{-1}(\frac{\partial}{\partial z}(\infty))$. But these facts may be checked on a dense open subset of S (or T), and so they follow from Proposition 3.6, Equations (17) and (18).

Claim II. The composition $U' \to T \to U'$ is the identity.

To prove this claim, we may pass to a dense open substack of U', because the target U' is separated. We start with an S-valued point (C, f, ϕ) of U', which defines a prestable marked curve (C, x). If we assume that S is étale over U', then the locus $S' \subset S$ over which (C, x) is stable is dense in S. Over S', neither stabilization nor blowing up changes (C, x) at all, so that, at least over S', the curve (C, x) agrees with the one obtained by applying the composition $U' \to T \to U'$. To check that also the map f does not change after passing through $U' \to T \to U'$, we may make S' still smaller, and apply the uniqueness part of Proposition 3.6. $\square$

Remark Denote by U_a the open substack of $\overline{M}_{0,0}(\mathbb{P}^1, d)$ consisting of maps which are unramified over $a \in \mathbb{P}^1$. Obviously, U_a is isomorphic to U_∞ and hence $U_a \cong T$, for all a. Choosing any N distinct points $a_1, \ldots, a_N \in \mathbb{P}^1$, where $N > 2d-2$, let $U_i = U_{a_i}$. Then $U_1, \ldots, U_N$ cover $\overline{M}_{0,0}(\mathbb{P}^1, d)$. Thus it is possible to obtain $\overline{M}_{0,0}(\mathbb{P}^1, d)$ by gluing together N copies of $[T/S_d]$. Ultimately, this leads to a complete description of the stable map stack $\overline{M}_{0,0}(\mathbb{P}^1, d)$ in terms of the stable curve space $\overline{M}_{0,d}$.

In this way, many questions about $\overline{M}_{0,0}(\mathbb{P}^1, d)$ can be reduced to questions about $\overline{M}_{0,d}$.

The action of the multiplicative group

We endow $\mathbb{P}^1$ with the right action of the group G (see Remark 1.2) given in Example 1.3. As usual, we get an induced (right) action of B on $\overline{M}_{0,0}(\mathbb{P}^1, d)$ given by

$$(C, f)(a, \lambda) = \big(C, (a, \lambda) \circ f \big) .$$

As in Remark 1.2, this means that we have a $\mathbb{G}_m$-action and an equivariant vector field on $\overline{M}_{0,0}(\mathbb{P}^1, d)$.

Since $\infty \in \mathbb{P}^1$ is a fixed point for the $\mathbb{G}_m$-action on $\mathbb{P}^1$, it is obvious that the stack of stable maps unramified over ∞ is invariant under the $\mathbb{G}_m$-action on $\overline{M}_{0,0}(\mathbb{P}^1, d)$. In this section we will determine the induced $\mathbb{G}_m$-action on T. Since the stack of maps unramified over ∞ is not invariant under all of B, we cannot describe an induced action of B on T, but, of course, we can pull back the equivariant vector field to T. Since this is more involved, we shall postpone it to Section 4.

Proposition 3.9 *Let $\mathbb{G}_m$ act on T through scalar multiplication on the vector bundle T over $\overline{M}_{0,d}$. This $\mathbb{G}_m$-action commutes with the S_d-action, hence induces a $\mathbb{G}_m$-action on $[T/S_d]$. Then the open immersion (25) of Theorem 3.7 is $\mathbb{G}_m$-equivariant.*

PROOF. Let $t \in T$. We have to show that for $\lambda \in \mathbb{C}^*$ we have

$$(C_t, \lambda \circ f_t) \cong (C_{\lambda t}, f_{\lambda t})$$

as stable maps. But to check that two automorphisms of T agree, we may pass to a dense open subscheme of T. So we may assume that $C_t = C_{\lambda t} = \mathbb{P}^1$ and that $f_t : \mathbb{P}^1 \to \mathbb{P}^1$ is given by (19). Then the linearity of (19) in $(b, q_1, \ldots, q_d)$, which are coordinates on the fibers of $T \to \overline{M}_{0,d}$, implies that $\lambda \circ f_t = f_{\lambda t}$, which implies the claim. $\square$

3.4 Maps to $\mathbb{P}^n$

Let $Y \subset \mathbb{P}^n$ be the open subvariety defined by

$$Y = \{ \langle x_0, \ldots, x_n \rangle \in \mathbb{P}^n \mid x_0 \neq 0 \text{ or } x_1 \neq 0 \} .$$

The map $\langle x_0, \ldots, x_n \rangle \mapsto \langle x_0, x_1 \rangle$ defines a morphism $p : Y \to \mathbb{P}^1$.

$$
\begin{array}{ccc}
Y & \hookrightarrow & \mathbb{P}^n \\
\downarrow{\scriptstyle p} & & \\
\mathbb{P}^1 & &
\end{array}
$$

Let us denote the fiber of p over ∞ by Y_∞. Thus $Y_\infty = \{\langle 0, 1, y_2, \ldots, y_n \rangle \in \mathbb{P}^n\}$, and Y_∞ is canonically identified with $\mathbb{A}^{n-1}$.

We will describe the open substack $U \subset \overline{M}_{0,0}(\mathbb{P}^n, d)$ consisting of stable maps (C, f) such that $f(C) \subset Y$ and $(C, p \circ f)$ is unramified over $\infty = \langle 0, 1 \rangle \in \mathbb{P}^1$. In fact, we will show that U is a vector bundle over $\overline{M}_{0,d}$ modulo the action of S_d.

Let $U_1 \subset \overline{M}_{0,0}(\mathbb{P}^1, d)$ be the open substack of stable maps unramified over ∞. By definition, we have a cartesian diagram

$$
\begin{array}{ccc}
U & \hookrightarrow \overline{M}_{0,0}(Y, d) \hookrightarrow & \overline{M}_{0,0}(\mathbb{P}^n, d) \\
\downarrow & \quad \square \qquad \downarrow & \\
U_1 & \hookrightarrow \overline{M}_{0,0}(\mathbb{P}^1, d) &
\end{array}
$$

Let

$$
T = \mathbb{A}^1 \times \left(\prod_{i=1}^d L_i \right) \times \left(\mathbb{A}^1 \times \mathbb{A}^d \right)^{n-1}, \tag{26}
$$

where the product is taken over $\overline{M}_{0,d}$. We will write an element of $T(S)$, for a scheme S, as

$$
(C, x, b, \tau, r)
$$

where $(C, x) = (C, x_1, \ldots, x_d) \in \overline{M}_{0,d}(S)$ is a stable marked curve over S, $b = (b_1, \ldots, b_n)$ is an n-tuple of regular functions on S, $\tau = (\tau_1, \ldots, \tau_d)$, where τ_i is a section of L_i over S and $r = (r_{\nu,i})_{\substack{\nu=2,\ldots,n \\ i=1,\ldots,d}}$ is a $d(n-1)$-tuple of regular functions on S. The order of the various coordinates in (26) is

$$
\left(b_1, (\tau_i), b_2, (r_{2,i}), \ldots, b_n, (r_{n,i}) \right).
$$

The reason for this order will become clear below.

For notational convenience, we assign the value 1 to $r_{1,i}$, for all $i = 1, \ldots, d$.

For $\nu = 1, \ldots, n$ consider

$$
b_\nu + \sum_{i=1}^d h_i(r_{\nu,i} \tau_i) \in \Gamma\left(C_T, \mathcal{O}(\textstyle\sum_i D_i)\right).
$$

Let, as above, $Z = Z_1 \cup \ldots \cup Z_d$, where $Z_i = D_i \cap \{\tau_i = 0\}$. After blowing up $Z \subset C_T$ we get a morphism

$$
f = \langle 1, \varphi_1, \ldots, \varphi_n \rangle : \widetilde{C} \longrightarrow Y,
$$

where

$$\varphi_\nu = b_\nu + \sum_{i=1}^{d} \widetilde{h}_i(r_{\nu,i}\widetilde{\tau}_i) = b_\nu + \sum_{i=1}^{d} r_{\nu,i} f_i \,. \tag{27}$$

This follows easily from the fact that already the sections 1 and $\varphi_1 = b_1 + \sum_i \widetilde{h}_i(\widetilde{\tau}_i) = b_1 + \sum_i f_i$ of $\mathcal{O}(\sum_i \widetilde{D}_i)$ generate this invertible sheaf, so that any values for $\varphi_2, \dots, \varphi_n$ define a morphism to $\mathbb{P}^n$. But, for the same reason, this morphism factors through Y.

Thus $(\widetilde{C}, f)$ is a stable map of degree d parametrized by T and hence we get a morphism

$$T \longrightarrow U \subset \overline{M}_{0,0}(\mathbb{P}^n, d) \,. \tag{28}$$

We will show that (28) induces an isomorphism $[T/S_d] \cong U$.

Let us write $r_i = \langle 0, 1, r_{2i}, \dots, r_{ni} \rangle$. Then we can say that $r_1, \dots, r_d \in Y_\infty$ are the points where our stable map intersects Y_∞ and τ_i gives the 'speed' with which the map passes through the point r_i. The coordinate b_ν gives the average of the ramification points of the projection onto the ν-th coordinate axis.

The next proposition makes this more precise.

Proposition 3.10 *For any $i = 1, \dots, d$ the composition $f \circ x_i$ defines a morphism*

$$f \circ x_i : T \longrightarrow Y_\infty \,,$$

which is given by $f \circ x_i = r_i$.

Fix a value of $\nu = 2, \dots, n$. Over the locus where $(r_{\nu,1}, \dots, r_{\nu,d}) \neq 0$ we can compose f with the projection p_ν onto the coordinate axis $\{\langle y_0, 0, \dots, y_\nu, \dots, 0 \rangle\}$ to obtain a stable map $f_\nu = p_\nu \circ f$ unramified over $\infty = \langle 0, \dots, 1, \dots, 0 \rangle$. Let $R_\nu \subset \widetilde{C}$ be the ramification scheme of f_ν. Then we have

$$b_\nu = \frac{1}{2d - 2} \operatorname{tr}_{R_\nu/T}(f|R_\nu) \,.$$

PROOF. To check that the two morphism $f \circ x_i$ and r_i agree, we may restrict to the locus where none of the τ_i or $r_{\nu,i}$ vanish. Then we need to prove that

$$\left(\frac{\varphi_2(s)}{\varphi_1(s)}, \dots, \frac{\varphi_n(s)}{\varphi_1(s)} \right) \bigg|_{s=x_i} = (r_{2,i}, \dots, r_{n,i}) \,.$$

This follows from (17), using l'Hôpital's rule. The second claim, giving the meaning of the b_ν, follows directly from (18). $\square$

Let $T_1 \to \overline{M}_{0,0}(\mathbb{P}^1, d)$ be the scheme constructed in Section 3.3 and called simply T there. By forgetting the new coordinates we introduced here, we get a morphism $T \to T_1$, which makes T into a vector bundle over T_1. Note that

$$\begin{array}{ccc} T & \longrightarrow & \overline{M}_{0,0}(Y, d) \\ \downarrow & & \downarrow \\ T_1 & \longrightarrow & \overline{M}_{0,0}(\mathbb{P}^1, d) \end{array}$$

commutes. So letting $\widetilde{U}$ be the fibered product $\widetilde{U} = T_1 \times_{U_1} U$ we get the diagram

$$
\begin{array}{ccccccc}
T & \longrightarrow & \widetilde{U} & \longrightarrow & U & \hookrightarrow & \overline{M}_{0,0}(Y,d) & \hookrightarrow & \overline{M}_{0,0}(\mathbb{P}^n, d) \\
& \searrow & \downarrow & \square & \downarrow & \square & \downarrow \\
& & T_1 & \longrightarrow & U_1 & \hookrightarrow & \overline{M}_{0,0}(\mathbb{P}^1, d)
\end{array}
$$

By Proposition 2.1 the scheme $\widetilde{U}$ is also a vector bundle over T_1.

Proposition 3.11 *The morphism $T \to \widetilde{U}$ is an isomorphism of vector bundles over T_1. Hence, $[T/S_d] \cong U$.*

PROOF. To check that $T \to \widetilde{U}$ is a morphism of vector bundles is made easier by the fact that it suffices for this to prove compatibility with the (linear) $\mathbb{G}_m$-actions. This is proved directly from the definitions of the two vector bundle structures using Proposition 3.10.

Since both vector bundles have the same rank $(d+1)(n-1)$, for the isomorphism property it suffices to prove strict injectivity, i.e., injectivity over every base change to a point of T_1. Then the claim also follows easily from Proposition 3.10. $\square$

Proposition 3.12 *Let $\mathbb{G}_m$ act on T by*

$$
\begin{aligned}
\big(C, x, b_1, (\tau_i), b_2, (r_{2,i}), \ldots, b_n, (r_{n,i})\big)\lambda \\
= \big(C, x, b_1\lambda, (\tau_i\lambda), b_2\lambda^2, (r_{2,i}\lambda), \ldots, b_n\lambda^n, (r_{n,i}\lambda^{n-1})\big).
\end{aligned}
$$

In other words, we let b_ν have weight ν, we let τ_i have weight 1 and we let $r_{\nu,i}$ have weight $\nu - 1$. Then $T \to \overline{M}_{0,0}(\mathbb{P}^n, d)$ is $\mathbb{G}_m$-equivariant.

PROOF. This also follows from Proposition 3.10 using the definition of the $\mathbb{G}_m$-action on $\mathbb{P}^n$ from Example 1.3. $\square$

3.5 The degree 2 case

Since $\overline{M}_{0,2}$ does not exist, the above considerations to not apply directly to the degree 2 case. We show how to treat this case here.

Let $U \subset \overline{M}_{0,0}(\mathbb{P}^n, 2)$ be, as above, the open substack of stable maps (C, f) such that

(i) $f(C) \subset Y$,

(ii) $p \circ f$ is unramified over $\infty \in \mathbb{P}^1$ (or f intersects Y_∞ transversally).

We will show how to write U as the stack quotient $[\mathbb{A}^{3n-1}/S_2]$, for a suitable S_2-action on $\mathbb{A}^{3n-1}$.

Let $T_1 = \mathbb{A}^2$, with coordinates b, q. Let $C = T_1 \times \mathbb{P}^1$, with sections $x_1 : T_1 \to T_1 \times \mathbb{P}^1; t \mapsto (t, 0)$ and $x_2 : T_1 \to T_1 \times \mathbb{P}^1; t \mapsto (t, \infty)$. Let s be the canonical affine coordinate on $\mathbb{P}^1$, $z = \frac{1}{s}$. Apply the program of Section 3.2 with b, $\tau_1 = \frac{\partial}{\partial s}(0)$

and $\tau_2 = q\frac{\partial}{\partial z}(\infty)$ to obtain a stable map of degree 2 from $\widetilde{C}$ to $\mathbb{P}^1$, unramified over ∞, where $\widetilde{C}$ is the blow up of $C = T_1 \times \mathbb{P}^1$ at $Z = Z_2 = D_2 \cap \{q = 0\}$. Note that even though (C, x_1, x_2) is not a stable curve, $(\widetilde{C}, x_1, \widetilde{x}_2, f)$ is a stable map. Over $T_1 - Z$, the morphism f is given by

$$f(s) = b + \frac{1}{s} + qs \,.$$

Let S_2 act trivially on T_1, and on $T_1 \times \mathbb{P}^1$ via the involution

$$\sigma : T_1 \times \mathbb{P}^1 \longrightarrow T_1 \times \mathbb{P}^1$$

$$(b, q, s) \longmapsto (b, q, \frac{1}{qs}) \,.$$

Note that σ extends uniquely to an automorphism $\widetilde{\sigma}$ of $\widetilde{C}$, giving rise to an action of S_2 on $\widetilde{C}$. Note also that $f \circ \widetilde{\sigma} = f$, and so we get a stable map

$$[\widetilde{C}/S_2] \xrightarrow{\ f\ } \mathbb{P}^1$$
$$\downarrow$$
$$[T_1/S_2]$$

of degree 2. Thus we have a morphism $[T_1/S_2] \to \overline{M}_{0,0}(\mathbb{P}^1, 2)$.

Lemma 3.13 *This morphism* $[T_1/S_2] \to \overline{M}_{0,0}(\mathbb{P}^1, 2)$ *is an isomorphism onto the open substack* U. $\square$

Now let

$$T = T_1 \times (\mathbb{A}^1 \times \mathbb{A}^2)^{n-1} \,,$$

with additional coordinates $(b_2, r_{21}, r_{22}), \ldots, (b_n, r_{n1}, r_{n2})$. Let $\widetilde{C}_T = \widetilde{C} \times_{T_1} T$ and define $\varphi : \widetilde{C}_T \to \mathbb{P}^n$ by $\varphi = \langle 1, f, \varphi_2, \ldots, \varphi_n \rangle$, where

$$\phi_\nu(s) = b_\nu + r_{\nu 1}\frac{1}{s} + r_{\nu 2}\, qs \,,$$

for $\nu = 2, \ldots, n$.

Let S_2 act on T by fixing T_1 and all b_ν, and exchanging $r_{\nu 1}$ with $r_{\nu 2}$, for $\nu = 2, \ldots, n$. Finally, let S_2 act on $\widetilde{C}_T$ diagonally, and denote the corresponding involution of $\widetilde{C}_T$ by $\widetilde{\sigma}_T$. Then $\varphi \circ \widetilde{\sigma}_T = \varphi$ and so we get an induced stable map $\varphi : [\widetilde{C}_T/S_2] \to \mathbb{P}^n$ parametrized by $[T/S_2]$ and hence a morphism $[T/S_2] \to \overline{M}_{0,0}(\mathbb{P}^n, 2)$.

Theorem 3.14 *The morphism* $[T/S_2] \to \overline{M}_{0,0}(\mathbb{P}^n, 2)$ *is an isomorphism onto the open substack* U. *Moreover,*

$$\begin{array}{ccc} [T/S_2] & \longrightarrow & \overline{M}_{0,0}(Y, 2) \\ \downarrow & & \downarrow \\ [T_1/S_2] & \longrightarrow & \overline{M}_{0,0}(\mathbb{P}^1, 2) \end{array} \qquad (29)$$

is a pullback diagram of vector bundles.□

Moreover, (29) is $\mathbb{G}_m$-equivariant if we let $\mathbb{G}_m$ act on T by

$$\left(b, q, \ldots, (b_\nu, r_{\nu 1}, r_{\nu 2}), \ldots\right) \cdot \lambda = \left(b\lambda, q\lambda^2, \ldots, (b_\nu \lambda^\nu, r_{\nu 1}\lambda^{\nu-1}, r_{\nu 2}\lambda^{\nu-1}), \ldots\right).$$

4 The vector field

4.1 Some deformation theory

Recall that the group $G = \mathbb{G}_a \rtimes \mathbb{G}_m$ acts on $\mathbb{P}^n$, as described in Example 1.3. We get induced actions on the stable map stacks $\overline{M}_{0,0}(\mathbb{P}^n, d)$ and $\overline{M}_{0,1}(\mathbb{P}^n, d)$ by the usual formula $(C, x, f) \cdot g = (C, x, g \circ f)$. The latter stack we may interpret as the universal curve C over $\overline{M}_{0,0}(\mathbb{P}^n, d)$, and so we have a diagram

$$
\begin{array}{ccc}
C & \xrightarrow{\;f\;} & \mathbb{P}^n \\
{\scriptstyle \pi}\downarrow & & \\
\overline{M}_{0,0}(\mathbb{P}^n, d) & &
\end{array}
$$

of stacks with G-equivariant morphisms. If we differentiate the $\mathbb{G}_a$-actions, we obtain $\mathbb{G}_m$-equivariant vector fields W on $\mathbb{P}^n$, V on $\overline{M}_{0,0}(\mathbb{P}^n, d)$ and U on C. Of course, U maps to V and W under $D\pi$ and Df, respectively. We are concerned with finding V.

In this section we will show that V and U are determined uniquely, simply by the fact that $D\pi(U) = V$ and $Df(U) = W$. The same is then true for the étale $\overline{M}_{0,0}(\mathbb{P}^n, d)$-scheme T: we know that we have found $V|T$ if we can find a vector field U on $\widetilde{C}$ that simultaneously lifts $V|T$ and W.

Proposition 4.1 *The derivative $Df \colon \mathcal{T}_C \to f^*\mathcal{T}_{\mathbb{P}^n}$ induces an isomorphism of vector bundles*

$$\pi_*\mathcal{T}_C \longrightarrow \pi_* f^*\mathcal{T}_{\mathbb{P}^n}.$$

We will prove this proposition below.

Corollary 4.2 *We have a diagram*

$$
\begin{array}{ccc}
 & & \Gamma(\mathbb{P}^n, \mathcal{T}_{\mathbb{P}^n}) \\
 & & \downarrow {\scriptstyle f^*} \\
\Gamma(\widetilde{C}, \mathcal{T}_{\widetilde{C}}) & \xrightarrow[Df]{\sim} & \Gamma(\widetilde{C}, f^*\mathcal{T}_{\mathbb{P}^n}) \\
{\scriptstyle D\widetilde{\pi}}\downarrow & & \\
\Gamma(T, \mathcal{T}_T) \xrightarrow{\;\sim\;} & \Gamma(\widetilde{C}, \widetilde{\pi}^*\mathcal{T}_T)\,. &
\end{array}
$$

By inverting the isomorphisms we get a homomorphism

$$\Gamma(\mathbb{P}^n, \mathcal{T}_{\mathbb{P}^n}) \longrightarrow \Gamma(T, \mathcal{T}_T),$$

which maps W to V. □

To prove Proposition 4.1 we start by recalling a few general facts about tangent complexes.

Lemma 4.3 *Let*

$$
\begin{array}{ccc}
X & \xrightarrow{\ f\ } & Y \\
{\scriptstyle p}\downarrow & & \downarrow{\scriptstyle g} \\
W & \xrightarrow{\ q\ } & Z
\end{array}
$$

be a cartesian diagram of smooth stacks. Let $r \cong q \circ p \cong g \circ f$. Then the diagram

$$
\begin{array}{ccc}
r^*\mathcal{T}_Z & \xrightarrow{\ K.S.(g)\ } & f^*\mathcal{T}^\bullet_{Y/Z}[1] = \mathcal{T}^\bullet_{X/W}[1] \\
{\scriptstyle K.S.(q)}\downarrow & & \downarrow \\
p^*\mathcal{T}^\bullet_{W/Z}[1] = \mathcal{T}^\bullet_{X/Y}[1] & \xrightarrow{\hspace{4cm}} & \mathcal{T}^\bullet_X[1]
\end{array}
$$

anti-commutes in the derived category of $\mathcal{O}_X$-modules. Here 'K.S.' stands for 'Kodaira-Spencer' map.

PROOF. Note that the composition

$$r^*\mathcal{T}_Z \xrightarrow{\ K.S.(r)\ } \mathcal{T}_{X/Z}[1] \xrightarrow{\ \alpha\ } \mathcal{T}_X[1]$$

is zero. On the other hand, we have

$$\mathcal{T}_{X/Z}[1] = \mathcal{T}_{X/Y}[1] \oplus \mathcal{T}_{X/W}[1]$$

and under this decomposition we have $K.S.(r) = K.S.(q) \oplus K.S.(g)$. Applying α, we get

$$0 = \alpha \circ K.S.(r) = \alpha \circ K.S.(q) + \alpha \circ K.S.(g).$$

This is what we wanted to prove. □

Lemma 4.4 *Again, considering a cartesian diagram as in the previous lemma, there is a canonical homomorphism of distinguished triangles*

$$
\begin{array}{ccccccc}
f^*\mathcal{T}^\bullet_{Y/Z} & \longrightarrow & f^*\mathcal{T}^\bullet_Y & \longrightarrow & f^*g^*\mathcal{T}^\bullet_Z & \xrightarrow{\ K.S.(g)\ } & f^*\mathcal{T}^\bullet_{Y/Z}[1] \\
\downarrow & & \| & & {\scriptstyle K.S.(q)}\downarrow & & \downarrow \\
\mathcal{T}^\bullet_X & \xrightarrow{\ \beta\ } & f^*\mathcal{T}^\bullet_Y & \longrightarrow & \mathcal{T}^\bullet_{X/Y}[1] & \xrightarrow{\ -\beta[1]\ } & \mathcal{T}^\bullet_X[1].
\end{array}
$$

The upper triangle is the pullback under f of the distinguished triangle associated with g, the lower triangle is a shift of the distinguished triangle associated with f.

PROOF. We have to show that the three squares commute. The last one is Lemma 4.3.

In any cartesian diagram as the one under consideration, there is a homomorphism from the distinguished triangle for p to the pullback under f of the distinguished triangle for g. Similarly, for f and q. Picking out appropriate commutative squares from these homomorphisms of distinguished triangles proves the commutativity of the other two squares. $\square$

Consider the morphism of stacks

$$h : \overline{M}_{0,0}(\mathbb{P}^n, d) \longrightarrow \mathfrak{M}_{0,0}.$$

Here $\mathfrak{M}_{0,0}$ is the Artin stack of prestable curves of genus zero. The morphism h is given by forgetting the map, retaining only the prestable curve (and not stabilizing). The fiber of h over a prestable curve C is equal to an open subscheme of $\mathrm{Mor}(C, \mathbb{P}^n)$, the scheme of morphisms from C to $\mathbb{P}^n$.

Deformation theory for morphisms shows that $\mathrm{Mor}(C, \mathbb{P}^n)$ is smooth with tangent space $H^0(C, f^*\mathcal{T}_{\mathbb{P}^n})$ at the point $f : C \to \mathbb{P}^n$ of $\mathrm{Mor}(C, \mathbb{P}^n)$. It follows that h is smooth with relative tangent bundle

$$\mathcal{T}_{\overline{M}/\mathfrak{M}} = \pi_* f^* \mathcal{T}_{\mathbb{P}^n}.$$

We shall now apply Lemma 4.4 to the cartesian square

$$
\begin{array}{ccc}
C & \xrightarrow{\;\;\pi\;\;} & \overline{M}_{0,0}(\mathbb{P}^n, d) \\
\downarrow & & \downarrow{\scriptstyle h} \\
\mathfrak{C} & \xrightarrow{\;\;\widetilde{\pi}\;\;} & \mathfrak{M}_{0,0}
\end{array}
$$

where $\mathfrak{C} \to \mathfrak{M}_{0,0}$ is the universal curve over $\mathfrak{M}_{0,0}$. We obtain the homomorphism of distinguished triangles

$$
\begin{array}{ccccccc}
\pi^* \mathcal{T}_{\overline{M}/\mathfrak{M}} & \longrightarrow & \pi^* \mathcal{T}_{\overline{M}} & \longrightarrow & \pi^* h^* \mathcal{T}_{\mathfrak{M}}^\bullet & \longrightarrow & \pi^* \mathcal{T}_{\overline{M}/\mathfrak{M}}[1] \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathcal{T}_C & \longrightarrow & \pi^* \mathcal{T}_{\overline{M}} & \longrightarrow & \mathcal{T}_{C/\overline{M}}^\bullet[1] & \longrightarrow & \mathcal{T}_C[1]
\end{array}
$$

and by adjointness, the homomorphism

$$
\begin{array}{ccccccc}
\mathcal{T}_{\overline{M}/\mathfrak{M}} & \longrightarrow & \mathcal{T}_{\overline{M}} & \longrightarrow & h^* \mathcal{T}_{\mathfrak{M}}^\bullet & \longrightarrow & \mathcal{T}_{\overline{M}/\mathfrak{M}}[1] \\
\downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \phi'} & & \downarrow{\scriptstyle \phi''} & & \downarrow \\
R\pi_* \mathcal{T}_C & \longrightarrow & R\pi_* \pi^* \mathcal{T}_{\overline{M}} & \longrightarrow & R\pi_* \mathcal{T}_{C/\overline{M}}^\bullet[1] & \longrightarrow & R\pi_* \mathcal{T}_C[1].
\end{array}
$$

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By the projection formula we have $R\pi_*\pi^*\mathcal{T}_{\overline{M}} = \mathcal{T}_{\overline{M}} \otimes R\pi_*\mathcal{O}_C = \mathcal{T}_{\overline{M}}$. Therefore ϕ' is an isomorphism. We also have that $R\pi_*\mathcal{T}_{C/\overline{M}}^\bullet[1] = h^*R\widetilde{\pi}_*\mathcal{T}_{\mathfrak{C}/\mathfrak{M}}^\bullet[1]$. Thus ϕ'' may be viewed as the pullback under h of the Kodaira-Spencer homomorphism $\mathcal{T}_{\mathfrak{M}}^\bullet \to R\widetilde{\pi}_*\mathcal{T}_{\mathfrak{C}/\mathfrak{M}}^\bullet[1]$, which is an isomorphism by the deformation theory of prestable curves.

Since both ϕ' and ϕ'' are isomorphisms, so is $\phi : \mathcal{T}_{\overline{M}/\mathfrak{M}} \to R\pi_*\mathcal{T}_C$.

Finally, let us consider the composition of homomorphisms of vector bundles on C

$$\pi^*\mathcal{T}_{\overline{M}/\mathfrak{M}} = \mathcal{T}_{C/\mathfrak{C}} \longrightarrow \mathcal{T}_C \xrightarrow{Df} f^*\mathcal{T}_{\mathbb{P}^n}.$$

Again, by adjointness, we get an induced homomorphism in the derived category of $\overline{M}_{0,0}(\mathbb{P}^n, d)$

$$\mathcal{T}_{\overline{M}/\mathfrak{M}} \xrightarrow{\phi} R\pi_*\mathcal{T}_C \xrightarrow{\psi} R\pi_*f^*\mathcal{T}_{\mathbb{P}^n}.$$

We just saw that ϕ is an isomorphism, and the composition $\psi \circ \phi$ is an isomorphism by the deformation theory of morphisms. Therefore, ψ is an isomorphism. Since $R\pi_*f^*\mathcal{T}_{\mathbb{P}^n} = \pi_*f^*\mathcal{T}_{\mathbb{P}^n}$, we deduce that $R\pi_*\mathcal{T}_C = \pi_*\mathcal{T}_C$, and that we have an isomorphism of vector bundles

$$\pi_*\mathcal{T}_C \longrightarrow \pi_*f^*\mathcal{T}_{\mathbb{P}^n}$$

which is induced by the derivative $Df : \mathcal{T}_C \to f^*\mathcal{T}_{\mathbb{P}^n}$. This finishes the proof of Proposition 4.1.

4.2   The degree 3 case

We will now use Corollary 4.2 to determine the vector field V on T in the case of $d = 3$.

So let $d = 3$. Then $\overline{M}_{0,3} = \operatorname{Spec}\mathbb{C}$. The universal curve C over $\overline{M}_{0,3}$ is isomorphic to $\mathbb{P}^1$, with 3 marked points $x_1, x_2, x_3 \in \mathbb{P}^1$. The vector bundle T over $\overline{M}_{0,3} = \operatorname{Spec}\mathbb{C}$ is just a vector space. We have canonically

$$T = \big(\mathbb{C} \oplus L_1 \oplus L_2 \oplus L_3\big) \oplus \big(\mathbb{C} \oplus \mathbb{C}^3\big)^{(n-1)},$$

where $L_i = \mathcal{T}_{\mathbb{P}^1}(x_i)$ is the tangent space of $\mathbb{P}^1$ at a marked point.

Every vector space E has a canonical vector field on it, namely the vector field that takes the value $e \in E$ at the point $e \in E$. This makes sense, because every tangent space of E is canonically identified with E. If we choose a basis (e_i) for E, then this vector field is given by $\sum_i x_i \frac{\partial}{\partial x_i} = \sum_i x_i e_i$, where (x_i) are the coordinates on E induced by the basis (e_i). We denote the canonical vector field on L_i by τ_i, for $i = 1, 2, 3$. Because of the product decomposition of T, we can also think of the τ_i as canonical vector fields on T.

Recall that the first coordinate of every quadruple of coordinates on T is called b_ν, where $\nu = 1, \dots, n$. The latter three coordinates in the latter $n-1$ quadruples of coordinates are called $(r_{\nu,i})_{\nu=2,\dots,n,\, i=1,2,3}$. Recall also that we write $r_{1,i} = 1$, $i = 1, 2, 3$, for notational convenience. For further notational convenience we write $r_{\nu,0} = b_\nu$, for $\nu = 1, \dots, n$. This then defines $r_{\nu,i}$ for

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all $\nu = 1,\dots,n$ and all $i = 0,1,2,3$. For every $\nu = 1,\dots,n$ we combine $r_{\nu,0}\dots,r_{\nu,3}$ into the column vector r_ν. Thus

$$r_1 = \begin{pmatrix} b_1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \quad \text{and} \quad r_\nu = \begin{pmatrix} b_\nu \\ r_{\nu,1} \\ r_{\nu,2} \\ r_{\nu,3} \end{pmatrix} \quad \text{for } \nu = 2,\dots,n$$

Because of the product decomposition of T, every vector field on T is in a canonical way a sum of $4n$ components. All but three are canonically identified with regular functions on T. We combine groups of four components into one as follows.

$$V_1 = V_{1,0}\frac{\partial}{\partial b_1} + \sum_{i=1}^{3} V_{1,i} \,,$$

and for $\nu = 2,\dots,n$

$$V_\nu = V_{\nu,0}\frac{\partial}{\partial b_\nu} + \sum_{i=1}^{3} V_{\nu,i}\frac{\partial}{\partial r_{\nu,i}} \,.$$

It will be convenient to introduce the following abbreviations:

$$\eta_i^{(2)} = -\prod_{j\neq i}\eta_{ij}$$

and for all $\mu \geq 1$

$$\theta_i^{(\mu)} = \sum_{j\neq i}\eta_{ij}^{\mu-1}\eta_{ji} \,,$$

(so that the upper indices denote the degree). Here the η_{ij} are the canonical regular functions on T introduced in (23). Finally, let

$$\ell_i = r_{2,i} - 2b_1 - 4\theta_i^{(1)} \,,$$

for $i = 1,2,3$.

Proposition 4.5 *With this notation, the vector field V on T is given by*

$$V_1 = r_2 - Er_1 + \begin{pmatrix} 0 \\ \ell_1\tau_1 \\ \ell_2\tau_2 \\ \ell_3\tau_3 \end{pmatrix} = \begin{pmatrix} b_2 - b_1^2 + \sum_i \eta_i^{(2)} + 5\sum_i \theta_i^{(2)} \\ \ell_1\tau_1 \\ \ell_2\tau_2 \\ \ell_3\tau_3 \end{pmatrix} ,$$

$$V_\nu = r_{\nu+1} - Er_\nu \,, \quad \text{for } \nu = 2,\dots,n-1 \,,$$
$$V_n = -Er_n \,.$$

Here E is the 4×4 matrix

$$E = \begin{pmatrix} b_1 & \eta_1^{(2)} + 5\theta_1^{(2)} & \eta_2^{(2)} + 5\theta_2^{(2)} & \eta_3^{(2)} + 5\theta_3^{(2)} \\ 1 & \ell_1 + b_1 + 3\theta_1^{(1)} & \eta_{21} & \eta_{31} \\ 1 & \eta_{12} & \ell_2 + b_1 + 3\theta_2^{(1)} & \eta_{32} \\ 1 & \eta_{13} & \eta_{23} & \ell_3 + b_1 + 3\theta_3^{(1)} \end{pmatrix} .$$

In the case $n = 1$, the formula for V_1 is used, not the formula for V_n. Moreover, b_2 and $r_{2,i}$ are set equal to 0 in this case.

PROOF. It suffices to check these formulas on a dense open subset of T. So we shall restrict to the subscheme of T where none of the τ_i vanish, so that we may assume that $\widetilde{C} = C_T = T \times \mathbb{P}^1$. Then all we have to do is exhibit a vertical vector field $\overline{U}$ for the projection $T \times \mathbb{P}^1 \to T$ such that $Df(V + \overline{U}) = W$. Then the vector field U on $\widetilde{C}$ is given by $U = V + \overline{U}$, over this open subscheme of $\widetilde{C}$.

The relative vector field $\overline{U}$ is very easy to describe. For every point t of T, we have given 3 tangent vectors on $\mathbb{P}^1$, namely the $\tau_i(t) \in \mathcal{T}_{\mathbb{P}^1}(x_i)$. On $\mathbb{P}^1$ there is a unique vector field $\overline{U}(t)$ taking this prescribed value $\tau_i(t)$ at the point x_i, for $i = 1, 2, 3$. This defines $\overline{U}$.

The details follow below. □

Lemma 4.6 *Let T be a scheme and $\tau_i \in \Gamma(T, L_i)$, for $i = 1, 2, 3$ a section of L_i over T. Then the formula*

$$\overline{U} = \sum_{i=1}^{3} \left(\prod_{j \neq i} \left(1 - \frac{s_i}{s_i(x_j)} \right) \right) \tau_i ds_i(x_i) \frac{\partial}{\partial s_i} \, ,$$

defines a relative vector field $\overline{U} \in \Gamma(T \times \mathbb{P}^1, \mathcal{T}_{T \times \mathbb{P}^1 / T}) = \Gamma(T \times \mathbb{P}^1, \mathcal{T}_{\mathbb{P}^1})$, which has the property that $\overline{U}(x_i) = \tau_i$, for $i = 1, 2, 3$. Here s_i is a degree one meromorphic function on $\mathbb{P}^1$, which vanishes at x_i (i.e., a parameter at x_i).

Let $f_i = h_i(\tau_i)$, for $i = 1, 2, 3$ be the meromorphic function on $T \times \mathbb{P}^1$ considered above (see Corollary 3.2). Then we have

$$\overline{U}(f_i) = 2\theta_i^{(2)} + \eta_i^{(2)} + 2\theta_i^{(1)} f_i - f_i^2 \, ,$$

for every $i = 1, 2, 3$.

PROOF. It is not difficult to see that the formula for $\overline{U}$ defines a vector field on $\mathbb{P}^1$, no matter the choice of the s_i, even if $s_i(x_j) = \infty$, for some $j \neq i$. The verification that $\overline{U}(x_i) = \tau_i$ is then easy.

The calculation of $\overline{U}(f_i)$ is somewhat tedious. It can be done by choosing a coordinate s for $\mathbb{P}^1$, which does not take the value ∞ at any of the x_i. Then write $a_i = s(x_i)$ and $s_i = s - a_i$. Finally, write $q_i = \tau_i ds_i(x_i)$. With these choices we have

$$\overline{U} = \sum_{i=1}^{3} q_i \left(\prod_{j \neq i} \frac{s - a_j}{a_i - a_j} \right) \frac{\partial}{\partial s}$$

and

$$f_i = q_i \left(\frac{1}{s - a_i} - \frac{1}{2} \sum_{j \neq i} \frac{1}{a_j - a_i} \right) \, .$$

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Thus

$$\overline{U}(f_i) = -\sum_{k=1}^{3} q_k \left(\prod_{j \neq k} \frac{s - a_j}{a_k - a_j} \right) \frac{q_i}{(s - a_i)^2}$$

and this can be compared to the given formula for $\overline{U}(f_i)$. $\square$

Lemma 4.7 *With the same notation, we have for $i \neq j$*

$$f_i f_j = \eta_{ji} f_i + \eta_{ij} f_j + 3\eta_{ij} \eta_{ji} \,.$$

In particular, summing over all i and j such that $i \neq j$,

$$\sum_{i \neq j} f_i f_j = 2 \sum_{i \neq j} \eta_{ji} f_i + 3 \sum_{i \neq j} \eta_{ij} \eta_{ji} \,.$$

PROOF. This can be proved similarly. $\square$

In view of the formula for W given in Example 1.3, the following corollary now finishes the proof of Proposition 4.5.

Corollary 4.8 *If we were to define a vector field V on T by the formulas of Proposition 4.5, then we would have*

$$V(\varphi_\nu) + \overline{U}(\varphi_\nu) = \varphi_{\nu+1} - \varphi_1 \varphi_\nu \,,$$

for $\nu < n$ and

$$V(\varphi_n) + \overline{U}(\varphi_n) = -\varphi_1 \varphi_n \,.$$

Here the φ_ν are the components of the morphism $f : \widetilde{C} \to \mathbb{P}^n$ given as in (27) by $\varphi_\nu = b_\nu + \sum_{i=1}^{3} r_{\nu,i} f_i$.

PROOF. We note that $\tau_j(f_i) = \delta_{ij} f_i$ (Kronecker delta). Moreover, $V(f_i) = \ell_i f_i$. Therefore, we need to prove the following formulas. First

$$V_{1,0} + \sum_i \ell_i f_i + \sum_i \overline{U}(f_i) = b_2 + \sum_i r_{2,i} f_i - \left(b_1 + \sum_i f_i \right)^2 \,,$$

then for every $1 < \nu < n$

$$V_{\nu,0} + \sum_i r_{\nu,i} \ell_i f_i + \sum_i V_{\nu,i} f_i + \sum_i r_{\nu,i} \overline{U}(f_i)$$
$$= b_{\nu+1} + \sum_i r_{\nu+1,i} f_i - \left(b_1 + \sum_i f_i \right) \left(b_\nu + \sum_i r_{\nu,i} f_i \right) \,,$$

and finally,

$$V_{n,0} + \sum_i r_{n,i}\ell_i f_i + \sum_i V_{n,i} f_i + \sum_i r_{n,i}\overline{U}(f_i)$$
$$= -\Big(b_1 + \sum_i f_i\Big)\Big(b_n + \sum_i r_{n,i} f_i\Big).$$

All of these formulas follow easily from Lemmas 4.6 and 4.7. $\square$

To make the relationship between vector fields and regular functions on the L_i explicit, we identify the universal curve C with $\mathbb{P}^1$, with affine coordinate s. Then we choose for x_1, x_2 and x_3, the points 0, 1 and ∞. We choose local parameters $\frac{1}{2}s$ at 0, $\frac{s-1}{2s}$ at 1 and $\frac{1}{2-2s}$ at ∞. This trivializes L_1, L_2 and L_3, and we write, as usual, the induced coordinates on L_i as q_i. With this notation the universal map is given by

$$\varphi_\nu(s) = b_\nu + r_{\nu,1}\, q_1 \frac{2-s}{s} + r_{\nu,2}\, q_2 \frac{s+1}{s-1} + r_{\nu,3}\, q_3(1-2s),$$

for all $\nu = 1, \dots, n$. Moreover, we have $\tau_i = q_i$ and

$$\eta_{12} = q_1 \qquad\qquad\qquad \eta_{13} = -q_1$$
$$\eta_{21} = -q_2 \qquad\qquad\qquad \eta_{23} = q_2$$
$$\eta_{31} = q_3 \qquad\qquad\qquad \eta_{32} = -q_3.$$

The matrix E then takes the following shape:

$$E = \begin{pmatrix} b & q_1\big(q_1 - 5(q_2+q_3)\big) & q_2\big(q_2 - 5(q_1+q_3)\big) & q_3\big(q_3 - 5(q_1+q_2)\big) \\ 1 & \ell_1 + b + 3(q_3 - q_2) & -q_2 & q_3 \\ 1 & q_1 & \ell_2 + b + 3(q_1 - q_3) & -q_3 \\ 1 & -q_1 & q_2 & \ell_3 + b + 3(q_2 - q_1) \end{pmatrix},$$

where we have dropped the index on b_1.

Corollary 4.9 *We have*

$$\mathbb{H}^0(T, K_V^\bullet) = \mathbb{C}[b, q_1, q_2, q_3, \ell_1, \ell_2, \ell_3]/(q_1\ell_1, q_2\ell_2, q_3\ell_3, R_{n+1})$$

where R_{n+1} denotes the quadruple of relations given by the matrix equation

$$R_{n+1} = E^n r_1.$$

The degree of all generators b, q_i, ℓ_i is one. The degrees of the components of R_{n+1} are $n+1$, n, n and n, respectively. The group S_3 acts as follows: $\sigma b = b$, $\sigma q_i = \mathrm{sign}(\sigma)q_{\sigma(i)}$, $\sigma\ell_i = \ell_{\sigma(i)}$, for $\sigma \in S_3$. Note that S_3 acts by a similar pattern on the seven relations.

PROOF. We have

$$\mathbb{H}^0(T, K_V^\bullet) = \mathbb{C}[b, (q_i)_{i=1,2,3}, (r_{\nu,i})_{\nu\geq 2, i\geq 0}]/(V_{\nu,i}).$$

The first four relations, $V_1 = 0$, give the relations $(q_i\ell_i)_{i=1,2,3}$ and $r_2 = Er_1$. The latter is only one relation and can be used to eliminate b_2, but not the $r_{2,i}$, for $i = 1, 2, 3$. Then the other relations $V_\nu = 0$, for $2 \leq \nu < n$, recursively eliminate all $r_{\nu,i}$, for $\nu \geq 3$. The last quadruple of relations can then be expressed as $R_{n+1} = 0$. Finally, we have replaced the generators $r_{2,i}$ by ℓ_i, for $i = 1, 2, 3$. $\square$

4.3 Chern classes

Recall the diagram

$$
\begin{array}{ccc}
C & \xrightarrow{\ f\ } & \mathbb{P}^n \\
{\scriptstyle \pi}\big\downarrow & & \\
\overline{M}_{0,0}(\mathbb{P}^n, d) & &
\end{array}
$$

involving the universal curve and the universal map. For every $m > 0$ we consider the vector bundle

$$
E_m = \pi_* f^* \mathcal{O}(m)
$$

on the stable map stack $\overline{M}_{0,0}(\mathbb{P}^n, d)$. Our goal in this section is to show how to determine the characteristic classes $c_{\widetilde{V}}^Q \in \mathbb{H}^0(T, K_V^\bullet)$ of E_m (see Section 1.3).

We restrict ourself to the case that $d = 3$. First, let us show that E_m is trivial over T.

Lemma 4.10 *Let* $f_i = \widetilde{h}_i(\widetilde{\tau}_i)$, $i = 1, 2, 3$, *be the canonical sections of* $\mathcal{O}_{\widetilde{C}}(\sum_i \widetilde{D}_i)$ *constructed in Section 3. Via the canonical identification* $f^* \mathcal{O}_{\mathbb{P}^n}(1) = \mathcal{O}_{\widetilde{C}}(\sum_i \widetilde{D}_i)$ *(see the definition of* f *in Section 3.4) we consider the* f_i *as global sections of* $\pi_* f^* \mathcal{O}(1)|T$. *Then the family* $(1, f_1, f_2, f_3)$ *trivializes this bundle. Moreover, the family* $(1, f_1, f_2, f_3, \dots, f_1^m, f_2^m, f_3^m)$ *of global sections of* $\mathcal{O}_{\widetilde{C}}(\sum_i m\widetilde{D}_i)$ *trivializes the bundle* $E_n|T$. *Finally,* $(1, f_1, f_2, f_3, \dots, f_1^m, f_2^m, f_3^m)$ *is a* $\mathbb{G}_{\mathrm{m}}$-*eigenbasis and we have* $^\lambda f_i^\nu = \lambda^\nu f_i$, *for all* $i = 0, \dots, m$.

PROOF. We have to show that for every geometric point t of T the family $(1, f_1, f_2, f_3)$ induces a basis of the fiber $\big(\pi_* f^* \mathcal{O}(1)\big)(t) = H^0(\widetilde{C}_t, \mathcal{O}(\sum_i \widetilde{x}_i))$. But this is true, because $f_i(t)$ has a pole of order one at $\widetilde{x}_i$, by construction. The 'moreover' follows similarly. $\square$

Recall that the vector field W on $\mathbb{P}^n$ acts on $\mathcal{O}(m)$ (see Examples 1.11 and 1.15). Via Lemmas 1.17 and 1.18 we get an induced action $\widetilde{V}$ of V on E_m, for all m. Of course, all these actions $\widetilde{V}$ are $\mathbb{G}_{\mathrm{m}}$-equivariant. To find the characteristic class $c_{\widetilde{V}}^Q(E_m)|T$ (where Q is an invariant polynomial function on $(3m + 1) \times (3m + 1)$-matrices), we only need to compute the matrix of $\widetilde{V}$ with respect to the basis of $E_m|T$ given by Lemma 4.10, and then evaluate Q on this matrix. This matrix can be computed from the following lemma, using Lemmas 4.6 and 4.7.

Lemma 4.11 *Let* $\widetilde{V}$ *be the action of* V *on* E_m. *Then we have, for all* $\mu = 0, \dots, m$

$$
\widetilde{V}(f_i^\mu) = (mb + \mu \ell_i) f_i^\mu + \mu f_i^{\mu-1} \overline{U}(f_i) + m f_i^{\mu+1} + m \sum_{m \neq i} f_i^\mu f_j \,.
$$

PROOF. It suffices to check this formula on a dense open subset of T. So we may assume that $\widetilde{C} = C = T \times \mathbb{P}^1$, and that the vector field U on $\widetilde{C}$ splits as $U = V + \overline{U}$, where $\overline{U}$ is the relative vector field defined in Lemma 4.6. Denote by $\widetilde{U}$ the action of U on $f^*\mathcal{O}(m)$. Then we have, by Lemmas 1.17 and 1.18

$$
\begin{aligned}
\widetilde{V}(f_i^\mu) &= \widetilde{U}(f_i^\mu) \\
&= \widetilde{U}(f_i^\mu 1) \\
&= U(f_i^\mu)1 + f_i^\mu \widetilde{U}(1) \\
&= \mu f_i^{\mu-1} U(f_i) + f_i^\mu \widetilde{U}(f^* x_0^m) \\
&= \mu f_i^{\mu-1} V(f_i) + \mu f_i^{\mu-1} \overline{U}(f_i) + f_i^\mu f^*\big(\widetilde{W}(x_0^m)\big) \\
&= \mu f_i^\mu \ell_i + \mu f_i^{\mu-1} \overline{U}(f_i) + f_i^\mu f^*\big(m x_0^{m-1} \widetilde{W}(x_0)\big)
\end{aligned}
$$

because, as we noted in the proof of Corollary 4.8, we have $V(f_i) = \ell_i f_i$,

$$
\begin{aligned}
&= \mu \ell_i f_i^\mu + \mu f_i^{\mu-1} \overline{U}(f_i) + f_i^\mu f^*(m x_0^{m-1} x_1) \\
&= \mu \ell_i f_i^\mu + \mu f_i^{\mu-1} \overline{U}(f_i) + m f_i^\mu f^*(x_1) \\
&= \mu \ell_i f_i^\mu + \mu f_i^{\mu-1} \overline{U}(f_i) + m f_i^\mu \big(b + \sum_j f_j\big) \\
&= \mu \ell_i f_i^\mu + \mu f_i^{\mu-1} \overline{U}(f_i) + m b f_i^\mu + m f_i^{\mu+1} + m \sum_{j \neq i} f_i^\mu f_j \,,
\end{aligned}
$$

which is what we wanted to prove. $\square$

Corollary 4.12 *The matrix of $\widetilde{V}$ acting on E_m, with respect to the basis $(1, f_1, f_2, f_3, \ldots, f_1^m, f_2^m, f_3^m)$ is determined by*

$$
\begin{aligned}
\widetilde{V}(1) &= mb + m(f_1 + f_2 + f_3)\,, \\
\widetilde{V}(f_i) &= (3m+2)\theta_i^{(2)} + \eta_i^{(2)} \\
&\quad + m \sum_{j \neq i} \eta_{ij} f_j + \big(\ell_i + mb + (m+2)\theta_i^{(1)}\big) f_i + (m-1) f_i^2 \,,
\end{aligned}
$$

and for all $\mu \geq 2$,

$$
\widetilde{V}(f_i^\mu) = 3m\theta_i^{(\mu+1)} + m \sum_{j \neq i} \eta_{ij}^\mu f_j + 4m \sum_{\alpha=1}^{\mu-2} \theta_i^{(\mu-\alpha+1)} f_i^\alpha
$$

$$
+ \big((2\mu+4m)\theta_i^{(2)} + \mu\eta_i^{(2)}\big) f_i^{\mu-1} + \big(\mu\ell_i + mb + (m+2\mu)\theta_i^{(1)}\big) f_i^\mu + (m-\mu) f_i^{\mu+1} \,,
$$

PROOF. We just need to plug in the formula for $\overline{U}$ given in Lemma 4.6 and the formula

$$
f_i^\mu f_j = \eta_{ji} f_i^\mu + 4 \sum_{\alpha=1}^{\mu-1} \eta_{ij}^{\mu-\alpha} f_i^\alpha + \eta_{ij}^\mu f_j + 3\eta_{ij}^\mu \eta_{ji} \,,
$$

which holds for $i \neq j$ and $\mu \geq 1$, and is proved by induction from Lemma 4.7. $\square$

Remark Note that the matrix for $\widetilde{V}$ on E_1 is the same as the matrix E, above, determining V itself. This is due to the following phenomenon. We may interpret the vector bundle E_1 as an open substack of $\overline{M}_{0,0}(\mathbb{P}^{n+1}, d)$. That gives us two vector fields on E_1. One is the reinterpretation of the action $\widetilde{V}$ as a vector field on E_1 (see Remark 1.10), the other is the vector field V on $\overline{M}_{0,0}(\mathbb{P}^{n+1}, d)$, restricted to E_1. These two vector fields are almost, but not quite, equal.

Corollary 4.13 *We have*

$$c_1(E_m) = m(3m+1)b + \frac{1}{2}m(m+1)\sum \ell_i$$

PROOF. The first Chern class of E_m is the trace of the matrix of $\widetilde{V}$ on E_m. Also note that $\sum_i \theta_i^{(1)} = 0$. $\square$

Corollary 4.14 *Let M_m be the matrix of $\widetilde{V}$ described in Corollary 4.12. Then we have*

$$c_2(E_m) = \frac{1}{2}\big(c_1^2(E_m) - \mathrm{tr}(M_m^2)\big)$$

and

$$\mathrm{tr}(M_n^2) = m^2(3m+1)b^2 + m^2(m+1)b\sum \ell_i + \frac{1}{6}m(m+1)(2m+1)\sum \ell_i^2$$
$$+ m(3m+1)(3m+2)\big(\sum \eta_i^{(2)} + \sum \theta_i^{(2)}\big) + \frac{1}{3}m(m+1)(7m+2)\sum \ell_i \theta_i^{(1)}\,.$$

PROOF. We use the formula

$$\sum_i (\theta_i^{(1)})^2 = 2\sum_i \eta_i^{(2)} + \sum_i \theta_i^{(2)}$$

throughout. $\square$

4.4 The cohomology of $\overline{M}_{0,0}(\mathbb{P}^\infty, 3)$

Introduce coordinates (q_i) as in the discussion leading up to Corollary 4.9.

Let us denote by T_n the étale $\overline{M}_{0,0}(\mathbb{P}^n, 3)$-scheme constructed in Section 3. For $1 \le n < m$ we consider $\mathbb{P}^n$ as a subvariety of $\mathbb{P}^m$ as in Section 2.2. We get an induced commutative diagram

$$
\begin{array}{ccc}
T_m & \longrightarrow & \overline{M}_{0,0}(\mathbb{P}^m, 3) \\
\uparrow & & \uparrow \\
T_n & \longrightarrow & \overline{M}_{0,0}(\mathbb{P}^n, 3)
\end{array}
\tag{30}
$$

The vector field V on $\overline{M}_{0,0}(\mathbb{P}^m, 3)$ restricts to the vector field V on $\overline{M}_{0,0}(\mathbb{P}^n, 3)$, so there is no ambiguity if we use the same later V to denote the vector field on these two stacks.

The diagram (30) induces the commutative diagram of $\mathbb{C}$-algebras

$$
\begin{array}{ccc}
\mathbb{H}^0(T_m, K_V^\bullet) & \longleftarrow & \mathbb{H}^0\big(\overline{M}_{0,0}(\mathbb{P}^m, 3), K_V^\bullet\big) \\
\downarrow & & \downarrow \\
\mathbb{H}^0(T_n, K_V^\bullet) & \longleftarrow & \mathbb{H}^0\big(\overline{M}_{0,0}(\mathbb{P}^n, 3), K_V^\bullet\big)
\end{array}
\tag{31}
$$

In terms of the explicit description given in Corollary 4.9, the vertical arrow on the left hand side of (31) is given by the canonical map

$$
\mathbb{C}[b, q_1, q_2, q_3, \ell_1, \ell_2, \ell_3]/(q_1\ell_1, q_2\ell_2, q_3\ell_3, R_{m+1})
$$
$$
\longrightarrow \mathbb{C}[b, q_1, q_2, q_3, \ell_1, \ell_2, \ell_3]/(q_1\ell_1, q_2\ell_2, q_3\ell_3, R_{n+1})
$$

and is therefore visibly surjective and moreover an isomorphism on the part of degree less than n. The vertical arrow on the right hand side of (31) is an isomorphism in degrees less than n by the comparison with $H_{DR}\big(\overline{M}_{0,0}(\mathbb{P}^m, 3)\big) \rightarrow H_{DR}\big(\overline{M}_{0,0}(\mathbb{P}^n, 3)\big)$, which enjoys these properties by the results of Section 2.2.

Just as in Section 2.2, we may therefore define limit algebras as follows:

$$
\mathbb{H}^0(T_\infty, K_V^\bullet) = \bigoplus_p \varprojlim_n \mathbb{H}^0(T_n, K_V^\bullet)_p \, ,
$$

and

$$
\mathbb{H}^0\big(\overline{M}_{0,0}(\mathbb{P}^\infty, 3), K_V^\bullet\big) = \bigoplus_p \varprojlim_n \mathbb{H}^0\big(\overline{M}_{0,0}(\mathbb{P}^n, 3), K_V^\bullet\big)_p \, ,
$$

where the subscript p denotes the homogeneous component of degree p. As above, the limits stabilize as soon as $n > p$. For every n we have a commutative diagram of $\mathbb{C}$-algebras

$$
\begin{array}{ccc}
\mathbb{H}^0(T_\infty, K_V^\bullet) & \longleftarrow & \mathbb{H}^0\big(\overline{M}_{0,0}(\mathbb{P}^\infty, 3), K_V^\bullet\big) \\
\downarrow & & \downarrow \\
\mathbb{H}^0(T_n, K_V^\bullet) & \longleftarrow & \mathbb{H}^0\big(\overline{M}_{0,0}(\mathbb{P}^n, 3), K_V^\bullet\big)
\end{array}
\tag{32}
$$

Theorem 4.15 *The canonical morphism*

$$
H_{DR}\big(\overline{M}_{0,0}(\mathbb{P}^\infty, 3)\big) = \mathbb{H}^0\big(\overline{M}_{0,0}(\mathbb{P}^\infty, 3), K_V^\bullet\big)
$$
$$
\longrightarrow \mathbb{H}^0(T_\infty, K_V^\bullet) = \mathbb{C}[b, q_1, q_2, q_3, \ell_1, \ell_2, \ell_3]/(q_1\ell_1, q_2\ell_2, q_3\ell_3) \, , \tag{33}
$$

is injective. Moreover, we have

$$
H_{DR}\big(\overline{M}_{0,0}(\mathbb{P}^\infty, 3)\big) = \mathbb{C}[b, \sigma_1, \rho, \sigma_2, \tau, \sigma_3]/\big((\tau^2 - \rho\sigma_2), \tau\sigma_3, \rho\sigma_3\big) \, ,
$$

where, via (33), we have the following identifications:

$$
\sigma_1 = \ell_1 + \ell_2 + \ell_3 \, ,
$$

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$$\rho = q_1^2 + q_2^2 + q_3^2 - 2(q_1 q_2 + q_1 q_3 + q_2 q_3)\,,$$
$$\sigma_2 = \ell_1^2 + \ell_2^2 + \ell_3^2 - 2(\ell_1 \ell_2 + \ell_1 \ell_3 + \ell_2 \ell_3)\,,$$
$$\tau = \ell_1(q_3 - q_2) + \ell_2(q_1 - q_3) + \ell_3(q_2 - q_1)\,,$$
$$\sigma_3 = \ell_1 \ell_3 \ell_3\,.$$

Note that $\rho = \sum \eta_i^{(2)} + \sum \theta_i^{(2)}$ and $\tau = \sum \ell_i \theta_i^{(1)}$ are the functions that appeared in Corollary 4.14.

PROOF. Let A be the graded $\mathbb{C}$-subalgebra of $\mathbb{H}^0\big(\overline{M}_{0,0}(\mathbb{P}^\infty, 3), K_V^\bullet\big)$ generated by the Chern classes $c_1(E_1)$, $c_1(E_2)$, $c_2(E_1)$, $c_2(E_2)$, $c_2(E_3)$ and $c_3(E_1)$. Let B the the image of A in $\mathbb{H}^0(T_\infty, K_V^\bullet)$. Then using Corollaries 4.13 and 4.14 and a direct calculation of $c_3(E_1)$ one shows that B is generated by

$$b, \sigma_1, \rho, \sigma_2, \tau, \sigma_3 \in \mathbb{C}[b, q_1, q_2, q_3, \ell_1, \ell_2, \ell_3]/(q_1 \ell_1, q_2 \ell_2, q_3 \ell_3)\,.$$

Now in [9] it is proved that the limit

$$\lim_{n \to \infty} \dim H_{DR}^p\big(\overline{M}_{0,0}(\mathbb{P}^n, 3)\big)$$

exists, and is equal to the coefficient of t^p in (34). So from the purely algebraic Lemma 4.16, we conclude that

$$\begin{aligned}
\dim H_{DR}^p\big(\overline{M}_{0,0}(\mathbb{P}^\infty, 3)\big) &\geq \dim A_p \\
&\geq \dim B_p \\
&= \lim_{n \to \infty} \dim H_{DR}^p\big(\overline{M}_{0,0}(\mathbb{P}^n, 3)\big)\,.
\end{aligned}$$

Hence we must have equality throughout, which proves the theorem. $\square$

Lemma 4.16 *The homomorphism of graded $\mathbb{C}$-algebras*

$$\mathbb{C}[b, \sigma_1, \rho, \sigma_2, \tau, \sigma_3]/\big((\tau^2 - \rho \sigma_2), \tau \sigma_3, \rho \sigma_3\big)$$
$$\longrightarrow \mathbb{C}[b, q_1, q_2, q_3, \ell_1, \ell_2, \ell_3]/(q_1 \ell_1, q_2 \ell_2, q_3 \ell_3)$$

defined by the formulas of Theorem 4.15 is injective. Moreover, the Hilbert series of $\mathbb{C}[b, \sigma_1, \rho, \sigma_2, \tau, \sigma_3]/\big((\tau^2 - \rho \sigma_2), \tau \sigma_3, \rho \sigma_3\big)$ is given by

$$\frac{1 + t + 2t^2 + 2t^3 + 2t^4}{(1 - t)(1 - t^2)^2(1 - t^3)}\,. \tag{34}$$

PROOF. Even a computer can prove this lemma, using *Macaulay 2*, for example. See [10]. $\square$

Remark Using the results of Section 4.3 it is possible to express the characteristic classes of the bundles E_m in terms of the generators given in Theorem 4.15. For the first two Chern classes we get

$$c_1(E_m) = m(3m + 1)\, b + \frac{1}{2} m(m + 1)\, \sigma_1\,,$$

$$c_2(E_m) =$$

$$\frac{3}{2}m^3(3m+1)\,b^2 + \frac{3}{2}m^3(m+1)\,b\sigma_1 + \frac{1}{24}m(m+1)(3m^2+m-1)\,\sigma_1^2$$

$$-\frac{1}{2}m(3m+1)(3m+2)\,\rho - \frac{1}{24}m(m+1)(2m+1)\,\sigma_2 - \frac{1}{6}m(m+1)(7m+2)\,\tau\,.$$

Corollary 4.17 *For every finite value of n there exists a natural morphism of $\mathbb{C}$-algebras*

$$\mathbb{C}[b,\sigma_1,\rho,\sigma_2,\tau,\sigma_3]/\big((\tau^2-\rho\sigma_2),\tau\sigma_3,\rho\sigma_3\big) \longrightarrow H_{DR}\big(\overline{M}_{0,0}(\mathbb{P}^n,3)\big)\,, \qquad (35)$$

which is an isomorphism in degrees less than n.

Corollary 4.18 *For every n, the cohomology ring $H_{DR}\big(\overline{M}_{0,0}(\mathbb{P}^n,3)\big)$ is generated by the characteristic classes of the bundles E_m, $m \geq 1$ (In fact the first three E_m and the first three Chern classes suffice).*

PROOF. Here we use Remark 2.5, from which we know that (35) is an algebra epimorphism. $\square$

4.5   A conjectural presentation for finite n

Let A be the 5×5-matrix

$$A = \begin{pmatrix} b+\sigma_1 & 0 & 9\,\rho+\frac{1}{4}(\sigma_2-\sigma_1{}^2)+3\,\tau & 0 & \sigma_3 \\ 0 & b+\frac{1}{2}\sigma_1 & 4\,\rho+\frac{1}{2}\,\tau & \rho & -\frac{1}{2}\sigma_1\rho \\ 1 & 0 & b & 0 & 0 \\ 0 & 1 & 0 & b & 0 \\ 0 & 0 & 1 & 1 & b \end{pmatrix}$$

and G_1 the column vector

$$G_1 = \begin{pmatrix} b\,\sigma_1 + 18\,\rho + \frac{1}{2}(\sigma_2+\sigma_1^2) + 6\,\tau \\ 9\,\rho+\tau \\ 2\,b+\sigma_1 \\ b \\ 3 \end{pmatrix}\,.$$

Conjecture 4.19 *We conjecture that*

$$H_{DR}\big(\overline{M}_{0,0}(\mathbb{P}^n,3)\big) = \mathbb{C}[b,\sigma_1,\rho,\sigma_2,\tau,\sigma_3]/\big((\tau^2-\rho\sigma_2),\tau\sigma_3,\rho\sigma_3,G_{n+1}\big)\,,$$

where $G_{n+1} = A^n G_1$. Note that the degrees of the moving relations given by G_{n+1} are n, $n+1$, $n+1$, $n+2$ and $n+2$.

Lemma 4.20 *There exists a morphism of $\mathbb{C}$-algebras*

$$\mathbb{C}[b, \sigma_1, \rho, \sigma_2, \tau, \sigma_3]/\big((\tau^2 - \rho\sigma_2), \tau\sigma_3, \rho\sigma_3, G_{n+1}\big)$$
$$\longrightarrow \mathbb{C}[b, q_1, q_2, q_3, \ell_1, \ell_2, \ell_3]/(q_1\ell_1, q_2\ell_2, q_3\ell_3, R_{n+1}) \quad (36)$$

defined by the formulas of Theorem 4.15.

PROOF. Consider the 5×4-matrix H given by

$$\begin{pmatrix} \sigma_1 & \rho + \ell_1^2 - q_1\lambda_1 + 5\ell_1\theta_1 & \rho + \ell_2^2 - q_2\lambda_2 + 5\ell_1\theta_2 & \rho + \ell_3^2 - q_3\lambda_3 + 5\ell_1\theta_3 \\ 0 & 3\rho + \ell_1\theta_1 & 3\rho + \ell_2\theta_2 & 3\rho + \ell_3\theta_1 \\ 2 & \ell_1 + 2\theta_1 & \ell_2 + 2\theta_2 & \ell_3 + 2\theta_3 \\ 1 & \theta_1 & \theta_2 & \theta_3 \\ 0 & 1 & 1 & 1 \end{pmatrix}$$

where $\lambda_1 = \ell_3 - \ell_2$, $\lambda_2 = \ell_1 - \ell_3$, $\lambda_3 = \ell_2 - \ell_1$ and $\theta_1 = q_3 - q_2$, $\theta_2 = q_1 - q_3$, $\theta_3 = q_2 - q_1$. This matrix satisfies

$$AH \equiv HE \bmod (q_1\ell_1, q_2\ell_2, q_3\ell_3)$$
$$HR_1 \equiv G_1 \bmod (q_1\ell_1, q_2\ell_2, q_3\ell_3).$$

By induction, this implies

$$G_{n+1} \equiv HR_{n+1} \bmod (q_1\ell_1, q_2\ell_2, q_3\ell_3),$$

which implies the lemma. $\square$

We have the following evidence for our conjecture:

Proposition 4.21 *Conjecture 4.19 holds for $n \leq 5$.*

PROOF. The following two (purely algebraic) conjectures imply Conjecture 4.19.
   Conjecture 1. The Hilbert series of the graded ring

$$C_n = \mathbb{C}[b, \sigma_1, \rho, \sigma_2, \tau, \sigma_3]/\big((\tau^2 - \rho\sigma_2), \tau\sigma_3, \rho\sigma_3, G_{n+1}\big)$$

is given by

$$\frac{(1 - t^n)(1 - t^{n+1})}{(1 - t)(1 - t^2)^2(1 - t^3)}\Big[t^{2n+3}(2 + 2t + 2t^2 + t^3 + t^4) \quad (37)$$
$$-t^{n+1}(1 + 3t + 4t^2 + 4t^3 + 3t^4 + t^5)$$
$$+(1 + t + 2t^2 + 2t^3 + 2t^4)\Big] \quad .$$

Conjecture 2. Let B_n be the image of the ring morhpism (36). Then for all $p \leq n + 2$ the dimension of the graded piece of degree p of B_n is given by the coefficient of t^p in Formula (37).

Both conjectures can be verified algorithmically for any given value of n. The authors did this using *Macaulay 2* (see [10]) for $n \leq 5$. In fact, the calculations indicate that $p \leq n + 2$ is not the best possible bound.

It is proved in [9], that (34) is the Poincaré polynomial of $\overline{M}_{0,0}(\mathbb{P}^n, 3)$.

Thus, using Conjecture 2, the morphism of $\mathbb{C}$-algebras

$$H_{DR}\big(\overline{M}_{0,0}(\mathbb{P}^n, 3)\big) \longrightarrow B_n$$

is an isomorphism in degrees $p \leq n + 2$. Consider the epimorphism (36) $C_n \to B_n$. By Conjecture 1, it is also an isomorphism in degrees $p \leq n + 2$.

$$
\begin{array}{ccc}
 & & C_n \\
 & \nearrow & \downarrow \\
H_{DR}\big(\overline{M}_{0,0}(\mathbb{P}^n, 3)\big) & \longrightarrow & B_n
\end{array}
$$

Since all relations in C_n are of degrees $p \leq n + 2$, we see that $C_n \to B_n$ lifts as indicated in the diagram. Now that we have an algebra morphism $C_n \to H_{DR}\big(\overline{M}_{0,0}(\mathbb{P}^n, 3)\big)$, we note that it is surjective by Remark 2.5 and hence an isomorphism, because both rings have the same Hilbert series. $\square$

4.6 The degree 2 case

Recall that the coordinates on $T = \mathbb{A}^2 \times (\mathbb{A}^1 \times \mathbb{A}^2)^{n-1}$ are called $(b, q, \dots, b_\nu, r_{\nu 1}, r_{\nu 2}, \dots)$. Accordingly, the vector field V has components

$$V = V_b \frac{\partial}{\partial b} + V_q \frac{\partial}{\partial q} + \sum_{\nu=2}^{n} \left(\frac{\partial}{\partial b_\nu} \quad \frac{\partial}{\partial r_{\nu 1}} \quad \frac{\partial}{\partial r_{\nu 2}} \right) \cdot V_\nu \,,$$

where V_b and V_q are regular functions on T, and V_ν, for $\nu = 2, \dots, n$ is a column vector thereof.

Proposition 4.22 *With this notation, the vector field V on T is given by*

$$
\begin{aligned}
V_b &= b_2 - b^2 - 4q \,, \\
V_q &= q(r_{21} + r_{22} - 4b) \,, \\
V_\nu &= r_{\nu+1} - Er_\nu \,, \quad \textit{for } \nu = 2, \dots, n-1 \\
V_n &= -Er_n \,,
\end{aligned}
$$

where for $\nu = 2, \dots, n$

$$r_\nu = \begin{pmatrix} b_\nu \\ r_{\nu 1} \\ r_{\nu 2} \end{pmatrix}$$

and

$$E = \begin{pmatrix} b & 2q & 2q \\ 1 & r_{21} - b & 0 \\ 1 & 0 & r_{22} - b \end{pmatrix} .$$

PROOF. Let $T' \subset T$, be the locus over which we have not blown up, so that the universal curve over T' is equal to $T' \times \mathbb{P}^1$. It suffices to exhibit a vertical vector field $\overline{U} \in \Gamma(T' \times \mathbb{P}^1, \mathcal{T}_{\mathbb{P}^1})$, such that, if we define V by the formulas of the proposition, then $Df(V + \overline{U}) = W$, where W is the vector field on $\mathbb{P}^n$ described in Example 1.3. This means that

$$V(\varphi_\nu) + \overline{U}(\varphi_\nu) = \varphi_{\nu+1} - \varphi_1 \varphi_\nu,$$

for $\nu < n$ and

$$V(\varphi_n) + \overline{U}(\varphi_n) = -\varphi_1 \varphi_n.$$

Here $\varphi_1 = b + \frac{1}{s} + qs$ and $\varphi_\nu = b_\nu + r_{\nu 1}\frac{1}{s} + r_{\nu 2}qs$, for $\nu > 1$. One checks that

$$\overline{U} = \frac{\partial}{\partial s} + (2b - r_{21})s\frac{\partial}{\partial s} - qs^2\frac{\partial}{\partial s}$$

solves this problem. $\square$

Let us write $R_1 = \begin{pmatrix} b \\ 1 \\ 1 \end{pmatrix}$.

Corollary 4.23 *With this abbreviation, we have*

$$\mathbb{H}^0(T, K_V^\bullet) = \mathbb{C}[b, q, r_1, r_2]/\big(q(r_1 + r_2 - 4b), R_{n+1}\big),$$

where $R_{n+1} = E^n R_1$ and we have dropped the first index of the r_{2i}.

The degrees of b, r_1, r_2 are 1, the degree of q is 2 and the degrees of the three components of R_{n+1} are $n+1$, n and n. The symmetric group S_2 acts by switching r_1 and r_2 and switching the last two components of R_{n+1}.

Now we determine the Chern classes of the vector bundles $E_m = \pi_* f^* \mathcal{O}(m)$.

Proposition 4.24 *We have*

$$c_1(E_m) = -mb + \frac{1}{2}m(m+1)(r_1 + r_2),$$

and

$$c_2(E_m) = \frac{1}{2}\big(c_1(E_m)^2 - \operatorname{tr} M_m^2\big),$$

where

$$\operatorname{tr} M_m^2 = \frac{2}{3}m(m^2 + 6m + 2)b^2 - \frac{1}{3}m(m+1)(m+2)b(r_1 + r_2)$$
$$+ \frac{1}{6}m(m+1)(2m+1)(8q + r_1^2 + r_2^2).$$

PROOF. We can trivialize E_m over T with the basis $(1, f_1, f_2, \dots, f_1^m, f_2^m)$, where $f_1 \in \Gamma\big(\widetilde{C}_T, \mathcal{O}(\widetilde{x}_1)\big)$ and $f_2 \in \Gamma\big(\widetilde{C}_T, \mathcal{O}(\widetilde{x}_2)\big)$ are the meromorphic functions $f_1 = \frac{1}{s}$, and $f_2 = qs$. Then we have for the action of V on this basis:

$$\widetilde{V}(1) = m(b + f_1 + f_2),$$
$$\widetilde{V}(f_1^\mu) = (\mu + m)qf_1^{\mu-1} + ((m - 2\mu)b + \mu r_1)f_1^\mu + (m - \mu)f_1^{\mu+1},$$
$$\widetilde{V}(f_2^\mu) = (\mu + m)qf_2^{\mu-1} + ((m - 2\mu)b + \mu r_2)f_2^\mu + (m - \mu)f_2^{\mu+1}.$$

This leads directly to the formulas of the proposition. $\square$

We are now ready to compute cohomology rings. Let us begin with the case $n = \infty$. Denote by $\mathbb{H}^0(T_\infty, K_V^\bullet)$ the limit of $\mathbb{H}^0(T, K_V^\bullet)$ as n goes to ∞. Thus

$$\mathbb{H}^0(T_\infty, K_V^\bullet) = \mathbb{C}[b, q, r_1, r_2]/\big(q(r_1 + r_2 - 4b)\big).$$

Proposition 4.25 *The canonical morphism*

$$\mathbb{H}^0\big(\overline{M}_{0,0}(\mathbb{P}^\infty, 2), K_V^\bullet\big) \longrightarrow \mathbb{H}^0(T_\infty, K_V^\bullet) \tag{38}$$

is injective. We have

$$H_{DR}\big(\overline{M}_{0,0}(\mathbb{P}^\infty, 2)\big) = \mathbb{C}[b, t, k],$$

where, under (38), we have $b \mapsto b$, $t \mapsto r_1 + r_2 - 2b$ and $k \mapsto 4q - (b - r_1)(b - r_2)$.

PROOF. This is proved the same way as Theorem 4.15. One uses that the Betti numbers of $\overline{M}_{0,0}(\mathbb{P}^n, 2)$ stabilize to the coefficients of

$$\frac{1}{(1 - t)^2(1 - t^2)}$$

(see [9]). $\square$

Corollary 4.26 *For every finite n, the cohomology ring of $\overline{M}_{0,0}(\mathbb{P}^n, 2)$ is generated by the Chern classes of the bundles E_m. In fact, $c_1(E_1)$, $c_1(E_2)$ and $c_2(E_1)$ will suffice.*

PROOF. It follows from Remark 2.5 that the cohomology ring of $\overline{M}_{0,0}(\mathbb{P}^n, 2)$ is generated by b, t and k, which translates easily into the claim about Chern classes, by virtue of the formulas of Proposition 4.24. $\square$

Now we are ready to determine the cohomology ring of $\overline{M}_{0,0}(\mathbb{P}^n, 2)$ for finite n.

Proposition 4.27 *We have*

$$H_{DR}\big(\overline{M}_{0,0}(\mathbb{P}^n, 2)\big) = \mathbb{C}[b, t, k]/(G_{n+1}).$$

Here G_{n+1} stands for three relations in degrees n, $n+1$ and $n+2$, defined recursively by the matrix equation $G_{n+1} = A^n G_1$, where

$$A = \begin{pmatrix} b & 0 & 0 \\ 1 & 0 & k \\ 0 & 1 & t \end{pmatrix}$$

and

$$G_1 = \begin{pmatrix} b(2b - t) \\ 2b - t \\ 2 \end{pmatrix} .$$

PROOF. The Poincaré polynomial of $\overline{M}_{0,0}(\mathbb{P}^n, 2)$, which we glean from [9] is

$$\frac{(1 - t^n)(1 - t^{n+1})(1 - t^{n+2})}{(t - 1)^2(1 - t^2)} . \tag{39}$$

Now the proof proceeds in the same way as the proof of Proposition 4.21, except for that we can prove the two requisite facts. They follows as lemmas. $\square$

In the following, we will abbreviate the three components of G_{n+1} as u_{n+1}, v_{n+1} and w_{n+1}. In particular, $u_{n+1} = b^{n+1}(2b - t)$. Similarly, the three components of R_{n+1} are x_{n+1}, y_{n+1} and z_{n+1}.

Lemma 4.28 *The Hilbert series of the graded ring $\mathbb{C}[b, t, k]/(G_{n+1})$ is given by (39).*

PROOF. First note that the only common zero of the three components of G_{n+1} is 0. One way to prove this follows. Assume that $u_{n+1} = 0$, $v_{n+1} = 0$ and $w_{n+1} = 0$. By the explicit description $b^\mu(2b - t)$ of u_μ we see that we get $u_\mu = 0$, for all $\mu \geq 1$. Then we have $v_{\mu+1} = kw_\mu$, for all μ, and the assumption that $k \neq 0$, forces $w_n = 0$. The conjunction of $w_{n+1} = 0$ and $w_n = 0$ forces $v_n = 0$. Proceeding inductively, we see that $w_1 = 0$, which contradicts the fact that $w_1 = 2$. Thus we can conclude that $k = 0$. Then all $v_\nu = 0$ and we have $w_{\mu+1} = tw_\mu$, and so $w_\mu = 2t^{\mu-1}$. This implies that $t = 0$. Finally, $u_{n+1} = 0$ then implies that $b = 0$.

We conclude that $\dim \mathbb{C}[b, t, k]/(G_{n+1}) = 0$, which implies that G_{n+1} is a regular sequence. Thus we can use the Koszul complex of G_{n+1} to compute the Hilbert series of this quotient ring. We get (39). $\square$

Lemma 4.29 *The formulas of Proposition 4.25 define a $\mathbb{C}$-algebra morphism*

$$\mathbb{C}[b, t, k]/(G_{n+1}) \longrightarrow \mathbb{C}[b, q, r_1, r_2]/\big(q(r_1 + r_2 - 4b), R_{n+1}\big) .$$

This morphism is injective in degrees $\leq n + 2$.

PROOF. The matrix

$$H = \begin{pmatrix} 4b - r_1 - r_2 & 0 & 0 \\ 2 & b - r_2 & b - r_1 \\ 0 & 1 & 1 \end{pmatrix}$$

satisfies $HE \equiv AH \bmod q(r_1 + r_2 - 4b)$ and $HR_1 = G_1$. By induction, we conclude that $HR_{n+1} \equiv G_{n+1} \bmod q(r_1 + r_2 - 4b)$. This proves the existence of the morphism. It is injective in degrees less than n, because $\mathbb{C}[b, t, k] \to \mathbb{C}[b, q, r_1, r_2]/q(r_1+r_2-4b)$ is injective. It remains to prove injectivity in degrees n, $n+1$ and $n+2$.

The invariant subring will be helpful. It is given by

$$\big(\mathbb{C}[b, q, r_1, r_2]/\big(q(r_1 + r_2 - 4b), R_{n+1}\big)\big)^{S_2}$$
$$= \mathbb{C}[b, t, k, q]/\big(q(2b - t), x_{n+1}, v_{n+1}, w_{n+1}\,.$$

The proof that this is indeed the ring of invariants proceeds as follows. Fist note that the invariant subring of $\mathbb{C}[b, q, r_1, r_2]$ is the polynomial ring $\mathbb{C}[b, t, k, q]$. Then, using the abbreviation $B = \mathbb{C}[b, q, r_1, r_2]$, we consider the exact sequence

$$B^4 \longrightarrow B \longrightarrow B/q(r_1 + r_2 - 4b), x_{n+1}, y_{n+1}, z_{n+1} \longrightarrow 0$$

of B-modules. Taking invariants of this sequence, gives us an exact sequence of B^{S_2}-modules. Since the invariants of B^4 are generated by $(1, 0, 0, 0)$, $(0, 1, 0, 0)$, $(0, 0, 1, 1)$ and $(r_1 - r_2)(0, 0, 1, -1)$, we conclude that the invariant subring of $B/q(r_1 + r_2 - 4b), x_{n+1}, y_{n+1}, z_{n+1}$ is equal to $\mathbb{C}[b, t, k, q]/q(2b - t), x_{n+1}, (r_1 - r_2)(y_{n+1} - z_{n+1}), (y_{n+1} + z_{n+1})$. That this is equivalent to the above presentation, follows from $HR_{n+1} \equiv G_{n+1}$.

So we are now reduced to proving that

$$\mathbb{C}[b, t, k]/\big(b^{n+2}(2b - t), v_{n+1}, w_{n+1}\big)$$
$$\longrightarrow \mathbb{C}[b, t, k, q]/\big(q(2b - t), x_{n+1}, v_{n+1}, w_{n+1}\big)$$

is injective in degrees n, $n+1$ and $n+2$. In the presentation $\mathbb{C}[b, t, k, q]/\big(q(2b - t), x_{n+1}, v_{n+1}, w_{n+1}\big)$, we may replace x_{n+1} by $b^{n+1} + q\widetilde{x}_{n+1}$, where $\widetilde{x}_{n+1} \in \mathbb{C}[b, t, k]$ and is recursively defined by $\widetilde{x}_1 = 0$, $\widetilde{x}_{n+1} = b\widetilde{x}_n + 2w_n$. Once we have done this, the proof proceeds as follows.

Let $a \in \mathbb{C}[b, t, k]$ be an element mapping to zero in $\mathbb{C}[b, t, k, q]/\big(q(2b - t), b^{n+1} + q\widetilde{x}_{n+1}, v_{n+1}, w_{n+1}$. Thus

$$a = e(q)q(2b - t) + f(q)(b^{n+1} + q\widetilde{x}_{n+1}) + g(q)v_{n+1} + h(q)w_{n+1}\,. \tag{40}$$

We may assume that every term in this equation is homogeneous. Let us expand in powers of q. The constant term gives us

$$a = f_0 b^{n+1} + g_0 v_{n+1} + h_0 w_{n+1}\,.$$

If $\deg a = n$, then for degree reasons it follows that $f_0 = 0$, $g_0 = 0$ and $h_0 \in \mathbb{C}$. Thus $a \in (w_{n+1})$, and we are done.

If $\deg a = n + 1$, then for degree reasons $f_0 \in \mathbb{C}$. Let us consider the linear term of (40).

$$0 = e_0(2b - t) + f_0 \widetilde{x}_{n+1} .$$

Since $(2b - t)$ does not divide $\widetilde{x}_{n+1}$ (see below), this implies that $f_0 = 0$. Thus $a \in (v_{n+1}, w_{n+1})$, and we are done.

Finally, let $\deg a = n + 2$. Then $\deg f_0 = 1$. This time the linear term of (40) is

$$0 = e_0(2b - t) + f_0 \widetilde{x}_{n+1} + h_1 w_{n+1} ,$$

where $h_1 \in \mathbb{C}$. Since $\widetilde{x}_{n+1}$ does not divides w_{n+1} modulo $2b - t$, this implies that $h_1 = 0$. This, in turn, implies that $(2b - t)$ divides f_0. Hence $a \in (b^{n+1}(2b - t), v_{n+1}, w_{n+1})$, and we are done.

It remains to check our two claims to the effect that if $t = 2b$, then $\widetilde{x}_{n+1} \neq 0$ and does not divide w_{n+1}. These can easily be proved by further setting $k = 0$, to solve the recursions. $\square$

Remark 4.30 It might be instructive to consider the case $d = 1$ in this context. Recall that $\overline{M}_{0,0}(\mathbb{P}^n, 1)$ is the Grassmannian of lines in $\mathbb{P}^n$. We let $T = \mathbb{A}^{2n-2}$, with coordinates $(b_2, r_2, \ldots, b_n, r_n)$. The universal curve over T is $T \times \mathbb{P}^1$. Let s be an affine coordinate on $\mathbb{P}^1$, then the universal map $f : T \times \mathbb{P}^1 \to \mathbb{P}^n$ is given by

$$s \longmapsto \langle 1, s, b_2 + r_2 s, \ldots, b_n + r_n s \rangle .$$

The auxiliary vertical vector field $\overline{U}$ is $(b_2 + r_2 s - s^2)\frac{\partial}{\partial s}$. This leads to the vector field V on T in the following form:

$$V_\nu = \begin{pmatrix} b_{\nu+1} \\ r_{\nu+1} \end{pmatrix} - \begin{pmatrix} 0 & b_2 \\ 1 & r_2 \end{pmatrix} \begin{pmatrix} b_\nu \\ r_\nu \end{pmatrix} , \qquad \text{for } \nu = 2, \ldots, n-1$$

$$V_n = - \begin{pmatrix} 0 & b_2 \\ 1 & r_2 \end{pmatrix} \begin{pmatrix} b_n \\ r_n \end{pmatrix} .$$

Here V_ν is the column vector of functions on T defined by

$$V = \sum_{\nu=2}^{n} \left(\frac{\partial}{\partial b_\nu} \quad \frac{\partial}{\partial r_\nu} \right) \cdot V_\nu .$$

The only zero of the vector field V is the origin of T. Thus we have

$$H^*(\overline{M}_{0,0}(\mathbb{P}^n, 1), \mathbb{C}) = \mathbb{H}^0\big(\overline{M}_{0,0}(\mathbb{P}^n, 1), K_V^\bullet\big)$$
$$= \Gamma(T, \mathcal{O})/V\Gamma(T, \Omega) = \mathbb{C}[b_2, r_2]/R_{n+1} ,$$

where

$$R_{n+1} = \begin{pmatrix} 0 & b_2 \\ 1 & r_2 \end{pmatrix}^n \begin{pmatrix} 0 \\ 1 \end{pmatrix} .$$

Note that the degree of b_2 is 2 and the degree of r_2 is 1. To find the geometric significance of r_2 and b_2, we calculate the action of V on the vector bundles $E_m = p_{T_*} f^* \mathcal{O}(m)$ over T. A basis is provided by $(1, s, \ldots, s^m)$ and V acts by $\widetilde{V}(s^\mu) = \mu b_2 s^{\mu-1} + \mu r_2 s^\mu + (m - \mu) s^{\mu+1}$. This leads to

$$c_1(E_m) = \frac{1}{2} m(m+1) r_2 \,.$$

Moreover, $c_1(E_1) = r_2$ and $c_2(E_1) = -b_2$.

In this case, there is a nice interpretation of the relations R_{n+1} in the cohomology ring. On $\overline{M}_{0,0}(\mathbb{P}^n, 1)$ there is an exact sequence of vector bundles

$$0 \longrightarrow K \longrightarrow \mathcal{O}^{n+1} \longrightarrow E_1 \longrightarrow 0 \,,$$

simply obtained by pushing forward and pulling back the tautological sequence on $\mathbb{P}^n$. The relations R_{n+1} are equivalent to the equation

$$c_t(K) c_t(E_1) = 1$$

among total Chern classes, which follows from the above short exact sequence. This short exact sequence does not generalize to the cases of higher degree, and we know no such simple motivation of our relations.

References

[1] E. Akildiz and J.B. Carrell. Cohomology of projective varieties with reglular SL_2 actions. *Manuscripta Math.*, 58:473–486, 1987.

[2] K. Behrend. Localization and Gromov-Witten invariants. In *Quantum Cohomology (Cetraro, 1997)*, volume 1776 of *Lecture Notes in Mathematics*. Springer, 2002.

[3] K. Behrend and B. Fantechi. The intrinsic normal cone. *Invent. Math.*, 128(1):45–88, 1997.

[4] K. Behrend and Yu. Manin. Stacks of stable maps and Gromov-Witten invariants. *Duke Math. J.*, 85(1):1–60, 1996.

[5] A. Białynicki-Birula. Some theorems on actions of algebraic groups. *Ann. of Math. (2)*, 98:480–497, 1973.

[6] J.B. Carrell and D.I. Lieberman. Holomorphic vector fields and Kaehler manifolds. *Invent. Math.*, 21:303–309, 1973.

[7] J.B. Carrell and D.I. Lieberman. Vector fields and Chern numbers. *Math. Ann.*, 225:263–273, 1977.

[8] P. Deligne and D. Mumford. The irreducibility of the space of curves of given genus. *Inst. Hautes Études Sci. Publ. Math.*, 36:75–109, 1969.

[9] E. Getzler and R. Pandharipande. The Poincare Polynomial of Kontsevich's space of stable maps $\overline{\mathcal{M}}_{0,n}(\mathbb{P}^n, d)$. in preparation.

[10] D. Grayson and M. Stillman. Macaulay 2, a software system for research in algebraic geometry. Available at `http://www.math.uiuc.edu/Macaulay2`.

[11] A. Grothendieck. On the de Rham cohomology of algebraic varieties. *Inst. Hautes Études Sci. Publ. Math.*, 29:95–103, 1966.

[12] M. Kontsevich. Enumeration of rational curves via torus actions. In *The moduli space of curves (Texel Island, 1994)*, pages 335–368. Birkhäuser Boston, Boston, MA, 1995.

[13] M. Kontsevich and Yu. Manin. Gromov-Witten classes, quantum cohomology, and enumerative geometry. *Comm. Math. Phys.*, 164:525–562, 1994.