Forbidden Configurations: A Survey

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Introduction

Forbidden configurations are first described as a problem area in a 1985 paper. My subsequent work has involved a number of coauthors: Farzin Barekat, Laura Dunwoody, Ron Ferguson, Balin Fleming, Zoltan Füredi, Jerry Griggs, Nima Kamoosi, Steven Karp, Peter Keevash, Miguel Raggi and Attila Sali but there are works of other authors (some much older, some recent) impinging on this problem as well (e.g. Balachandran, Dukes). For example, the definition of *VC-dimension* uses a forbidden configuration. The notion of *trace* is the set theory name for a configuration.

Survey at www.math.ubc.ca/∼anstee

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i.e. if A is m-rowed then A is the incidence matrix of some family $\mathcal{A}$ of subsets of $[m] = \{1, 2, \ldots, m\}$.

$$A = \begin{bmatrix} 0 & 0 & 0 & \boxed{1} & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

$$\mathcal{A} = \{\emptyset, \{2\}, \{3\}, \{1,3\}, \{1,2,3\}\}$$

Definition Given a matrix F, we say that A has F as a *configuration* if there is a submatrix of A which is a row and column permutation of F.

$$F = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix} \quad \in \quad \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & \boxed{1 \quad 0 \quad 1} & & 1 & \boxed{0} \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & \boxed{1 \quad 1 \quad 0} & & 0 & \boxed{0} \end{bmatrix} = A$$

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We consider the property of forbidding a configuration F in A for which we say F is a *forbidden configuration* in A.

Definition Let $\mathrm{forb}(m, F)$ be the largest function of m and F so that there exist a $m \times \mathrm{forb}(m, F)$ simple matrix with *no* configuration F. Thus if A is any $m \times (\mathrm{forb}(m, F) + 1)$ simple matrix then A contains F as a configuration.

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For example, $\text{forb}(m, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}) = m + 1$

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Theorem (Sauer 72, Perles and Shelah 72, Vapnik and Chervonenkis 71)

$$forb(m, K_k) = \binom{m}{k-1} + \binom{m}{k-2} + \cdots + \binom{m}{0} = \Theta(m^{k-1})$$

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Theorem (Füredi 83). *Let F be a $k \times \ell$ matrix. Then* $forb(m, F) = O(m^k)$

Two interesting examples

$$\text{Let} \qquad F_1 = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \qquad F_2 = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$\text{forb}(m, F_1) = 2m, \qquad \text{forb}(m, F_2) = \left\lfloor \frac{m^2}{4} \right\rfloor + m + 1$$

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Problem *What drives the asymptotics of forb(m, F)? What structures in F are important?*

A Product Construction

The building blocks of our product constructions are I, I^c and T:

$$I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad I_4^c = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}, \quad T_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Note that

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} \notin I, \quad \begin{bmatrix} 0 \\ 0 \end{bmatrix} \notin I^c, \quad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \notin T$$

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Note that $\text{forb}(m, \begin{bmatrix} 1 \\ 1 \end{bmatrix}) = \text{forb}(m, \begin{bmatrix} 0 \\ 0 \end{bmatrix}) = \text{forb}(m, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}) = m + 1$

Definition Given an $m_1 \times n_1$ matrix A and a $m_2 \times n_2$ matrix B we define the product $A \times B$ as the $(m_1 + m_2) \times (n_1 n_2)$ matrix consisting of all $n_1 n_2$ possible columns formed from placing a column of A on top of a column of B. If A, B are simple, then $A \times B$ is simple. (A, Griggs, Sali 97)

$$\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix} \times \begin{bmatrix}1 & 1 & 1\\0 & 1 & 1\\0 & 0 & 1\end{bmatrix} = \begin{bmatrix}1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0\\0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0\\0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1\\1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1\\0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1\\0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1\end{bmatrix}$$

Given p simple matrices $A_1, A_2, \ldots, A_p$, each of size $m/p \times m/p$, the p-fold product $A_1 \times A_2 \times \cdots \times A_p$ is a simple matrix of size $m \times (m^p/p^p)$ i.e. $\Theta(m^p)$ columns.

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The Conjecture

Definition Let $x(F)$ denote the largest p such that there is a p-fold product which does not contain F as a configuration where the p-fold product is $A_1 \times A_2 \times \cdots \times A_p$ where each $A_i \in \{I_{m/p}, I^c_{m/p}, T_{m/p}\}$.

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The conjecture has been verified for $k \times \ell$ F where $k = 2$ (A, Griggs, Sali 97) and $k = 3$ (A, Sali 05) and $l = 2$ (A, Keevash 06) and for k-rowed F with bounds $\Theta(m^{k-1})$ or $\Theta(m^k)$ plus other cases.

Refinements of the Sauer Bound

Theorem (Sauer 72, Perles and Shelah 72, Vapnik and Chervonenkis 71) *forb*(m, K_k) *is* $\Theta(m^{k-1})$

Let $E_1 = \begin{bmatrix}1\\1\end{bmatrix}, E_2 = \begin{bmatrix}0\\0\end{bmatrix}, E_3 = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$.

Theorem (A, Fleming) *Let* F *be a* $k \times l$ *simple matrix such that there is a pair of rows with no configuration* E_1 *and there is a pair of rows with no configuration* E_2 *and there is a pair of rows with no configuration* E_3*. Then forb*(m, F) *is* $O(m^{k-2})$.

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Note that $F_1 = \begin{bmatrix}1 & 0 & 1 & 0\\0 & 1 & 1 & 1\\0 & 0 & 0 & 1\end{bmatrix}$ has no E_1 on rows 1,3, no E_2 on rows 1,2 and no E_3 on rows 2,3. Thus forb(m, F_1) is $O(m)$.

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Theorem (A, Fleming) *Let E be given with $E \in \{E_1, E_2, E_3\}$. Let F be a $k \times l$ simple matrix with the property that every pair of rows contains the configuration E. Then forb$(m, F) = \Theta(m^{k-1})$.*

$$F_2 = \begin{bmatrix}1 & 0 & 1 & 0\\0 & 1 & 0 & 1\\0 & 0 & 1 & 1\end{bmatrix}$$ has E_3 on rows 1,2.

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$$F_2 = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$ has E_3 on rows 2,3.

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Note that F_2 has E_3 on every pair of rows hence forb(m, F_2) is $\Theta(m^2)$ (A, Griggs, Sali 97).

In particular, this means $F_2 \notin T \times T$ which is the construction to achieve the bound.

Definition Let $t \cdot M$ be the matrix $[M\, M \cdots M]$ consisting of t copies of M placed side by side.

Theorem (A, Füredi 86) *Let k, t be given.*

$$forb(m, t \cdot K_k) \leq \frac{t-2}{k+1}\binom{m}{k} + \binom{m}{k} + \binom{m}{k-1} + \cdots \binom{m}{0}$$

with equality if a certain k-design exists

Definition Let $\mathbf{1}_k\mathbf{0}_\ell$ denote the vector of k 1's on top of ℓ 0's and let $\mathbf{1}_k$ be the vector of k 1's.

Theorem (A, Füredi 86) *Let k, t be given.*

$$forb(m, t \cdot K_k) = forb(m, t \cdot \mathbf{1}_k)$$

In order for a 4-rowed F to have $\text{forb}(m, F)$ be quadratic in m, we must have the associated simple matrix have a quadratic bound. There are three simple column-maximal 4-rowed F for which $\text{forb}(m, F)$ is quadratic. Here is one example:

$$F_3 = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{bmatrix}$$

How can we repeat columns in F_3 and still have a quadratic bound? We note that $\text{forb}(m, 2 \cdot \mathbf{1}_1\mathbf{0}_3)$ and $\text{forb}(m, 2 \cdot \mathbf{1}_3\mathbf{0}_1)$ are both cubic and so we only consider taking multiple copies of the columns of sum 2.

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$$F_3(t) = \left[\begin{matrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{matrix}\; t \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \\ 0 & 0 \end{bmatrix}\right]$$

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Theorem (A, Raggi, Sali 09) Let t be given. Then forb$(m, F_3(t))$ is $O(m^2)$.

Note that $F_3 = F_3(1)$. The proof is currently a rather complicated induction.

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There are six k-rowed F for each $k \geq 6$ to consider in order to establish the boundary between $O(m^{k-2})$ and $\Omega(m^{k-1})$ (using the predictions of the conjecture) and there are only three 4-rowed F to consider of which $F_3(t)$ is one. The other two have not been shown to have quadratic bounds.

Let

$$F_4(t) = t \cdot \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}$$

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This appears to be a pivotal case for 4-rowed F:

Theorem (A, Barekat, Sali 09)

$$\text{forb}(m, F_4(1)) = \binom{m}{2} + m - 2$$

The above conjecture follows from the general conjecture and is true for $t = 2$. Solving this would no doubt enable the proof of quadratic bounds for the two other 4-rowed F

Boundary cases k-rowed F with bounds $\theta(m^{k-1})$

Let B be a $k \times (k+1)$ matrix which has one column of each column sum. Given two matrices C, D, let $C \backslash D$ denote the matrix obtained from C by deleting any columns of D that are in C (i.e. set difference). Let

$F_B(t) = [K_k | t \cdot [K_k \backslash B]]$.

Theorem (A, Griggs, Sali 97, A, Sali 05, A, Fleming, Füredi, Sali 05)
Let t, B be given. Then forb$(m, F_B(t))$ is $\Theta(m^{k-1})$.

The difficult problem here was the bound although induction works.

Let D be the $k \times (2^k - 2^{k-2} - 1)$ simple matrix with all columns of sum at least 1 that do not simultaneously have 1's in rows 1 and 2. We take $F_D(t) = [\mathbf{0}_k \; (t+1) \cdot D]$ which for $k = 4$ becomes

$$F_D(t) = \left[\begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \end{array} \; (t+1) \cdot \left[\begin{array}{ccccccccccc} 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \end{array} \right] \right]$$

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The argument used standard results for directed graphs, *indicator polynomials* and a linear algebra rank argument

Theorem *Let k be given and assume F is a k-rowed configuration which is not a configuration in $F_B(t)$ for any choice of B as a $k \times (k+1)$ simple matrix with one column of each column sum and not in $F_D(t)$ or $F_D(t)^c$, for any t. Then forb(m, F) is $\Theta(m^k)$.*

Designs and Forbidden Configurations

A 2-design $S_\lambda(2,3,v)$ consists of $\frac{\lambda}{3}\binom{v}{2}$ triples from $[v]=\{1,2,\ldots,v\}$ such that for each pair $i,j\in\binom{[v]}{2}$, there are exactly λ triples containing i,j. If we encode the triple system as a v-rowed (0,1)-matrix A such that the columns are the incidence vectors of the triples, then A has no $2\times(\lambda+1)$ submatrix of 1's.

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Remark If A is a $v\times n$ (0,1)-matrix with column sums 3 and A has no $2\times(\lambda+1)$ submatrix of 1's then $n\le\frac{\lambda}{3}\binom{v}{2}$ with equality if and only if the columns of A correspond to the triples of a 2-design $S_\lambda(2,3,v)$.

Theorem (A, Barekat) Let λ and v be given integers. There exists an M so that for $v > M$, if A is an $v \times n$ (0,1)-matrix with column sums in $\{3, 4, \ldots, v-1\}$ and A has no $3 \times (\lambda+1)$ configuration

$$\begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ 0 & 0 & \cdots & 0 \end{bmatrix}$$

then

$$n \le \frac{\lambda}{3}\binom{v}{2}$$

and we have equality if and only if the columns of A correspond to the triples of a 2-design $S_\lambda(2, 3, v)$.

Theorem (A, Barekat) Let λ and v be given integers. There exists an M so that for $v > M$, if A is an $v \times n$ (0,1)-matrix with column sums in $\{3, 4, \ldots, v-3\}$ and A has no $4 \times (\lambda + 1)$ configuration

$$\begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \end{bmatrix}$$

then

$$n \leq \frac{\lambda}{3}\binom{v}{2}$$

with equality only if there are positive integers a, b with $a + b = \lambda$ and there are $\frac{a}{3}\binom{v}{2}$ columns of A of column sum 3 corresponding to the triples of a 2-design $S_a(2, 3, v)$ and there are $\frac{b}{3}\binom{v}{2}$ columns of A of column sum $v - 3$ corresponding to $(v - 3)$ - sets whose complements (in $[v]$) corresponding to the triples of a 2-design $S_b(2, 3, v)$.

Theorem (N. Balachandran 09) Let λ and v be given integers. There exists an M so that for $v > M$, if A is an $v \times n$ (0,1)-matrix with column sums in $\{4, 5, \ldots, v-1\}$ and A has no 4×2 configuration

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \\ 0 & 0 \end{bmatrix}$$

then

$$n \leq \frac{1}{4}\binom{v}{3}$$

with equality only if there is 3-design $S_1(3, 4, v)$. The $\frac{1}{4}\binom{v}{3}$ columns of A have column sum 4 and correspond to 4-sets of the 3-design $S_1(3, 4, v)$.

Naranjan Balachandran has indicated that he has made further progress on this problem

$k \times 2$ Forbidden Configurations

$$\text{Let } F_{abcd} = \begin{array}{l} a\left\{\begin{array}{l}\\ \\ \\ \end{array}\right. \\ b\left\{\begin{array}{l}\\ \\ \\ \end{array}\right. \\ c\left\{\begin{array}{l}\\ \\ \\ \end{array}\right. \\ d\left\{\begin{array}{l}\\ \\ \\ \end{array}\right. \end{array} \begin{bmatrix} 1 & 1 \\ \vdots & \vdots \\ 1 & 1 \\ 1 & 0 \\ \vdots & \vdots \\ 1 & 0 \\ 0 & 1 \\ \vdots & \vdots \\ 0 & 1 \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{bmatrix}$$

For the purposes of forbidden configurations we may assume that $a \geq d$ and $b \geq c$.

The following result used a difficult ‘stability’ result and the resulting constants in the bounds were unrealistic but the asymptotics are further evidence for the conjecture.

Theorem (A-Keevash 06) *Assume a,b,c,d are given with* $a \geq d$ *and* $b \geq c$. *If* $b > c$ *or* $a, b \geq 1$, *then*

$$forb(m, F_{abcd}) = \Theta(m^{a+b-1}).$$

Also $forb(m, F_{0bb0}) = \Theta(m^b)$ *and* $forb(m, F_{a00d}) = \Theta(m^a)$.

It is convenient to define $\mathbf{1}_k\mathbf{0}_\ell$ as the $(k+\ell)\times 1$ column of k 1's on top of ℓ 0's. Then the first column of F_{abcd} is $\mathbf{1}_{a+b}\mathbf{0}_{c+d}$.

Theorem (A, Karp 09) Let $a, b \geq 2$. Then

$$\text{forb}(m, F_{ab01}) = \text{forb}(m, \mathbf{1}_{a+b}\mathbf{0}_1) = \sum_{j=0}^{a+b-1} \binom{m}{j} + \sum_{j=m}^{m} \binom{m}{j}$$

$$\text{forb}(m, F_{ab10}) = \text{forb}(m, \mathbf{1}_{a+b}\mathbf{0}_1) = \sum_{j=0}^{a+b-1} \binom{m}{j} + \sum_{j=m}^{m} \binom{m}{j}$$

$$\text{forb}(m, F_{ab11}) = \text{forb}(m, \mathbf{1}_{a+b}\mathbf{0}_2) = \sum_{j=0}^{a+b-1} \binom{m}{j} + \sum_{j=m-1}^{m} \binom{m}{j}$$

Problem (A, Karp 09). Let a, b, c, d be given with a, b much larger than c, d. Is it true that $\text{forb}(m, F_{abcd}) = \text{forb}(m, \mathbf{1}_{a+b}\mathbf{0}_{c+d})$?

Problem (A, Karp 09). Let a, b, c, d be given with a, b much larger than c, d. Is it true that $\text{forb}(m, F_{abcd}) = \text{forb}(m, \mathbf{1}_{a+b}\mathbf{0}_{c+d})$?

We are asking when we can make the first column with $a + b$ 1's and $c + d$ 0's dominate the bound.

$$F_{2110} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Not all $k \times 2$ cases are obvious:

Theorem *Let c be a positive real number. Let A be an $m \times \left(c\binom{m}{2} + m + 2\right)$ simple matrix with no F_{2110}. Then for some $M > m$, there is an $M \times \left((c + \frac{2}{m(m-1)})\binom{M}{2} + M + 2\right)$ simple matrix with no F_{2110}.*

Critical Substructures

Definition A *critical substructure* of a configuration F is a minimal configuration F' contained in F such that

$$\text{forb}(m, F) = \text{forb}(m, F')$$

A critical substructure is what drives the construction yielding a lower bound $\text{forb}(m, F)$ where some other argument provides the upper bound for $\text{forb}(m, F)$.
A consequence is that for a configuration F'' which contains F' and is contained in F, we deduce that

$$\text{forb}(m, F) = \text{forb}(m, F'') = \text{forb}(m, F')$$

Critical Substructures for K_3

$$K_3 = \begin{bmatrix} 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Critical substructures are $\mathbf{1}_3$, K_3^2, K_3^1, $\mathbf{0}_3$, $2 \cdot \mathbf{1}_2$, $2 \cdot \mathbf{0}_2$ since $\text{forb}(m, \mathbf{1}_3) = \text{forb}(m, K_3^1) = \text{forb}(m, K_3^2) = \text{forb}(m, \mathbf{0}_3)$ $= \text{forb}(m, 2 \cdot \mathbf{1}_2) = \text{forb}(m, 2 \cdot \mathbf{0}_2)$.

Critical Substructures for K_3

$$K_3 = \left[\begin{array}{|c|ccccccc} \color{red}{1} & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ \color{red}{1} & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ \color{red}{1} & 0 & 1 & 1 & 0 & 0 & 1 & 0 \end{array}\right]$$

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Critical Substructures for K_3

$$K_3 = \left[\begin{array}{ccccccc|c|} 1 & 1 & 1 & 0 & 1 & 0 & 0 & \color{red}{0} \\ 1 & 1 & 0 & 1 & 0 & 1 & 0 & \color{red}{0} \\ 1 & 0 & 1 & 1 & 0 & 0 & 1 & \color{red}{0} \end{array}\right]$$

Critical substructures are $\mathbf{1}_3$, K_3^2, K_3^1, $\color{red}{\mathbf{0}_3}$, $2 \cdot \mathbf{1}_2$, $2 \cdot \mathbf{0}_2$ since $\text{forb}(m, \mathbf{1}_3) = \text{forb}(m, K_3^1) = \text{forb}(m, K_3^2) = \text{forb}(m, \mathbf{0}_3)$
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Critical Substructures for K_3

$$K_3 = \left[\begin{array}{cc|cccccc} 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ \hline 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \end{array}\right]$$

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Critical Substructures for K_3

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Critical substructures are $\mathbf{1}_3$, K_3^2, K_3^1, $\mathbf{0}_3$, $2 \cdot \mathbf{1}_2$, $2 \cdot \mathbf{0}_2$ since $\text{forb}(m, \mathbf{1}_3) = \text{forb}(m, K_3^1) = \text{forb}(m, K_3^2) = \text{forb}(m, \mathbf{0}_3)$ $= \text{forb}(m, 2 \cdot \mathbf{1}_2) = \text{forb}(m, 2 \cdot \mathbf{0}_2)$.

Another Example of Critical Substructures

$$F_5 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Theorem (A, Karp 09) For $m \geq 3$ we have

$$\text{forb}(m, F_5) = \text{forb}(m, 2 \cdot \mathbf{1}_2\mathbf{0}_1) = \text{forb}(m, 2 \cdot \mathbf{1}_1\mathbf{0}_2) = \binom{m}{2} + m + 2.$$

Thus for

$$F_6 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

we deduce that $\text{forb}(m, F_6) = \text{forb}(m, F_5) = \text{forb}(m, 2 \cdot \mathbf{1}_2\mathbf{0}_1)$
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Another Example of Critical Substructures

$$F_5 = \left[\begin{array}{cc|cc} 1 & 1 & \color{red}{1} & \color{red}{1} \\ 1 & 1 & \color{red}{0} & \color{red}{0} \\ 1 & 0 & 1 & 0 \\ 0 & 0 & \color{red}{0} & \color{red}{0} \end{array}\right]$$

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Exact Bounds

$$F_7 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

Theorem (A,Karp 09)

$$\text{forb}(m, F_7) = \text{forb}(m, 3 \cdot \mathbf{1}_2) \leq \frac{4}{3}\binom{m}{2} + m + 1$$

with equality for $m \equiv 1, 3 (\text{mod } 6)$.

Exact Bounds

$$F_7 = \left[\begin{array}{ccc} \boxed{\begin{matrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{matrix}} \\ 1 \quad 0 \quad 0 \end{array}\right]$$

Theorem (A,Karp 09)

$$\text{forb}(m, F_7) = \text{forb}(m, 3 \cdot \mathbf{1}_2) \le \frac{4}{3}\binom{m}{2} + m + 1$$

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THANKS! ITS FUN TO VISIT WATERLOO AGAIN!

Definition We say $\mathcal{F} \subseteq 2^{[m]}$ is t-intersecting if for every pair $A, B \in \mathcal{F}$, we have $|A \cap B| \geq t$.

Theorem (Ahlswede and Khachatrian 97)
Complete Intersection Theorem.
Let k, r be given. A maximum sized $(k\text{-}r)$-intersecting k-uniform family $\mathcal{F} \subseteq \binom{[m]}{k}$ is isomorphic to $\mathcal{I}_{r_1,r_2}$ for some choice $r_1 + r_2 = r$ and for some choice $G \subseteq [m]$ where $|G| = k - r_1 + r_2$ where
$\mathcal{I}_{r_1,r_2} = \{A \subseteq \binom{[m]}{k} \,:\, |A \cap G| \geq k - r_1\}$

This generalizes the Erdős-Ko-Rado Theorem (61).

Theorem (A-Keevash 06) Stability Lemma.
Let $\mathcal{F} \subseteq \binom{[m]}{k}$. *Assume that* $\mathcal{F}$ *is* $(k\text{-}r)$*-intersecting and*

$$|\mathcal{F}| \geq (6r)^{5r+7} m^{r-1}.$$

Then $\mathcal{F} \subseteq \mathcal{I}_{r_1,r_2}$ *for some choice* $r_1 + r_2 = r$ *and for some choice* $G \subseteq [m]$ *where* $|G| = k - r_1 + r_2$.

This result is for large intersections; we use it with a fixed r where k can grow with m.

Definition Let $F_{a,b,c,d}$ denote the $(a+b+c+d)\times 2$ matrix of a rows [11], b rows of [10], c rows of [01], and d rows of [00]. We assume $a \ge d$ and $b \ge c$.

Theorem (A-Keevash 06) *if* $b > c$ *or* $a, b \ge 1$, *then*

$$forb(m, F_{a,b,c,d}) = \Theta(m^{a+b-1}).$$

Also $forb(m, F_{0,b,b,0}) = \Theta(m^b)$ *and* $forb(m, F_{a,0,0,d}) = \Theta(m^a)$.

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Proof: The conjecture yields constructions. The proofs of the bounds make heavy use of the stability lemma in conjunction with induction.

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Proof: The conjecture yields constructions. The proofs of the bounds make heavy use of the stability lemma in conjunction with induction.

The theorem is further evidence for the conjecture.

e.g. Let A be a simple matrix with no $F_{0,3,2,0} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}$

Let $A = \begin{bmatrix} 0\,0 \cdots 0 & 1\,1 \cdots 1 \\ B_1 \quad B_2 & B_2 \quad B_3 \end{bmatrix}$ (the standard induction),
where B_2 is chosen to be all columns which are repeated after deleting row 1 of A.
Then $[B_1 B_2 B_3]$ is simple and has no $F_{0,3,2,0}$ and so by induction has at most $c(m-1)^2$ columns. B_2 is also simple and we verify that B_2 has at most cm columns and so by induction
$\text{forb}(m, F_{0,3,2,0}) \le c(m-1)^2 + cm \le cm^2$.

Given $A = \begin{bmatrix} 0\,0 \cdots 0 & 1\,1 \cdots 1 \\ B_1 \quad B_2 & B_2 \quad B_3 \end{bmatrix}$ with no $F_{0,3,2,0} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}$

then B_2 has no $F_{0,2,2,0} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}$ or $F_{0,3,1,0} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$

Our proof then uses the fact that if we only consider the columns of column sum k in B_2 as a set system, then using the fact that $F_{0,2,2,0}$ is forbidden we deduce that the k-uniform set system is $(k-1)$-intersecting. We then use our stability result to either determine the columns have a certain structure or that the bound is true because there are so few columns.