MATH 444 Runner's problem

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Intermediate Value Theorem

I was told this problem while running with Claude Belisle, a professor at Université Laval. It is published in a paper:

C. Belisle, Le Problème du coureur et son interprétation probabiliste, *Ann. Sci. Math. Québec*, **19**(1995), 1-8.

I originally gave this in MATH 184 as a motivation for the Intermediate Value Theorem. We consider the situation of a runner who finished a 12km race in 48 minutes. This is on average 4 minutes per km. The runner is interested in running in an upcoming 10km race and wonders if he can do it in 40 minutes.

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Question 2. Did he complete a 6 km race in exactly 24 minutes in a 6 km segment of his 12 km race?

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The answer to Question 2 is Yes and the answer to Question 1 is no (or in particular, not necessarily).

Intermediate Value Theorem. Let $f : [a, b] \to \mathbf{R}$ be a continuous function. Without loss of generality, assume $f(a) \leq f(b)$. Let $c \in [f(a), f(b)]$. Then there is some $x \in [a, b]$ with $f(x) = c$.

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For Question 2 we define a function $f(x)$ that considers the various 6 km segments that could be considered. We define
$f(x) =$ time required to run from $x$km to $x + 6$km segment of 12 km race.
Thus f has domain $[0, 6]$. We observe that f is continuous and so we can apply the Intermediate Value Theorem.
We consider $f(0)$ and $f(6)$. We note that $f(0) + f(6) = 48$ since the two 6 km segments covers the entire 12 km race. Now if $f(0) = 24$ we are done and have Yes answer. If $f(0) < 24$, then $f(6) > 24$ using $f(0) + f(6) = 48$. But now we may appeal to the Intermediate Value Theorem to deduce there is some $c \in [0, 6]$ with $f(c) = 24$. Similarly for $f(0) > 24$. Thus we have run the 6 km (from c km to $c + 6$ km) in 24 minutes answering Yes to Question 2

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This same idea can be applied to show there is some 4km segment of the 12km race in which the runner took exactly 16 minutes.

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For Question 1 the following (reasonable) running pattern demonstrates a case where the answer in No. Run the first km in 3 minutes, the next 10 kms in 42 minutes at a constant pace and the last km in 3 minutes. As for question 1, we may define a continuous function
$g(x) =$ time required to run from $x$km to $x + 10$km segment of 12 km race.
Thus g has domain $[0, 2]$. But for our given running pattern, $g(x) > 24$ for all x in the domain. e.g. $g(0) = 3 + 9 \times 4.2 = 40.8$ minutes. This yields No to Question 1 (or at least we cannot conclude Yes in general). In this particular case the running pattern of the 12km race is quite believable. It is temptingly easy to conclude (incorrectly) using average time of 4 min per km, there must be choices x, y with $g(x) \ge 40$ and $g(y) \le 40$.