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THE UNIVERSITY OF BRITISH COLUMBIA

Sessional Examination - December 2013

MATH 443: Graph Theory

Instructor: Dr. R. Anstee, section 101

Special Instructions: No Aids. No calculators or cellphones. Time: 3 hours

You must show your work and explain your answers.

1. (10 marks) Let C_1, C_2 be two different cycles of G with $E(C_1) \cap E(C_2) \neq \emptyset$. Show that there is a third cycle C_3 in G.

2. (10 marks) Let G be a simple graph on 34 vertices with 5 vertices of degree 3 and 9 vertices of degree 1 and 20 vertices of degree 2. Show that G is not connected.

3. (10 marks)

   a) Let G be a 4-regular connected graph. Show that G has no cut edge (A *cut edge* is an edge e for which $G \backslash e$ is disconnected).

   b) Let G be a connected cubic graph and let G have an cut edge $e = xy$. Show that any perfect matching of G (if one exists) must use the edge e.

4. (10 marks)

   a) Show that there is no cubic plane graph with exactly one face of size 7 and the rest of the faces have sizes 4 or 6.

   b) Let G be a connected plane graph with all faces of size at least 4. Show that $e(G) \leq 2n(G) - 4$.

   c) Let G be a simple connected plane graph with s faces of size 5 and t faces of size 6 and no other faces. Assume G is cubic (all degrees are 3). Determine s.

5. (10 marks)

   a) Let G be a d-regular graph with $\kappa'(G) \geq d$. Show that for each k (with $k \geq 1$) and each $S \subseteq V(G)$ with $|S| = k$, show that $G - S$ has at most k components.

   b) Show that if G is d-regular with $\kappa'(G) \geq d$ and $|V(G)|$ being even, then G has a perfect matching.

6. (10 marks) Assume the directed graph $D = (N, A)$ is strongly connected. Show that there is a directed spanning subgraph $D' = (N, A')$ with $A' \subseteq A$ satisfying that D' is strongly connected and $|A'| \leq 2|N|$.

7. (10 marks) Let T and T' be two edge disjoint spanning trees on the same set of n vertices. Let G be the union of T and T', namely G is the graph with $V(G) = V(T) = V(T')$ and $E(G) = E(T) \cup E(T')$.

   a) Show that $\kappa'(G) \geq 2$.

   b) Show that $\kappa'(G) < 4$.

   c) Choose a value for n and choose two spanning trees T , T' whose union G has $\kappa'(G) = 3$.

8. (10 marks) Let W be a closed walk in a simple graph G. Let H be the spanning subgraph of G consisting of one copy of each edge that was used an odd number of times in W. Prove that for each $v \in V(G)$, $d_H(v)$ is even.

9. (10 marks) Let G be a planar graph which is uniquely 4-colourable, namely every 4-colouring of the vertices is the same up to a renaming of the colours. Show that $e(G) = 3n(G) - 6$.

10. (10 marks) Let D be an orientation of the complete graph K_n. Assume that D is strongly connected. Thus D has a directed cycle. Show that D has a directed cycle using all the vertices (a spanning directed cycle).

total 100 marks

1. Let G be a connected graph with $2k$ vertices of odd degree and the rest have even degree. Show that G has a trail with at least $|E(G)|/k$ edges.

2. (10 marks) Show how any simple graph G can be decomposed into even closed trails (i.e. closed trails using an even number of vertices), paths whose endpoints are odd degree vertices in G and vertex disjoint odd cycles.

3. (10 marks) Let W be a closed walk in a simple graph G. Let H be the spanning subgraph of G consisting of those edges used an odd number of times in W. Prove that for each $v \in V(G)$, $d_H(v)$ is even. (Note: the spanning subgraph H could have no edges).

4. (10 marks) An undirected graph G is *orientable* if the edges can be oriented so the the resulting directed graph $D(G)$ is strongly connected. e.g. C_n is *orientable*.

   a) Assume G is a connected graph with a cut edge $e = xy$. Show that G is not orientable.

   b) Assume that G is a 2-edge connected graph on at least 4 vertices, then G is orientable.

5. (10 marks) Let G be a simple graph with $\Delta(G) = 2$ and G has 4 vertices of degree 1. Show that G is disconnected.

6. (10 marks) Let G be a graph with k components and $e(G) = n(G) - k$. Show that G has no cycles.

7. (10 marks)

   a) Let G be a simple graph with $\chi(G) = 3$. Show that there is a subset S of the vertices with $|S| \geq (2/3)|V(G)|$ such that the subgraph of G induced by the vertices of S is bipartite.

   b) Extend this to graphs with $\chi(G) = k$ and determine a large fraction of the vertices which induce a bipartite subgraph of G. Show the result is best possible for $G = K_n$.

8. (10 marks) Let G be a cubic simple graph. Assume that G has a Hamiltonian cycle. Show that we can decompose $E(G)$ into 3 perfect matchings. Does the same result hold if we merely assume G has a spanning subgraph regular of degree 2 (a 2-factor)?

9. (10 marks) Let G be a connected simple graph. Let G have a cut vertex v so that G can be thought of as the union of two connected subgraphs H, K which overlap on the single vertex v. Show that

$$\chi(G; k) = \frac{1}{k}\chi(H; k)\chi(K; k).$$