

Math 340 A Theorem of the Alternative

The duality theory can be used to develop many theorems associated with inequalities and equalities. Some of these theorems were developed and proved long before the duality theorems had been proved or indeed before linear programming had been developed. (Some can be used to derive the duality theorems and provide different proofs to those we have given). If you step back, you can see the Linear Algebra handles the equalities extremely well and probably LP's are your first time to deal with lots of inequalities.

Question: Does $Ax \leq \mathbf{b}$ have a solution $\mathbf{x} \geq \mathbf{0}$?

To answer this question, which has no initial reference to linear programming, we would, if we were mathematicians, try setting up an LP for which the answer to the question corresponds to some property of the LP. Consider the (primal) LP and its dual:

$$\begin{array}{rcc} \max & ? & \min \\ \text{primal:} & Ax \leq \mathbf{b}, & \text{dual:} \quad A^T \mathbf{y} \geq ? \\ & \mathbf{x} \geq \mathbf{0} & \mathbf{y} \geq \mathbf{0} \end{array}$$

I have left ?'s to denote choices yet to be made. Notice that the question has the yes answer if the primal LP is feasible. Thus it makes sense to choose the objective function to not distinguish feasible solutions and so we take $z = \mathbf{0} \cdot \mathbf{x}$. This yields:

$$\begin{array}{rcc} \max & \mathbf{0} \cdot \mathbf{x} & \min \\ \text{primal:} & Ax \leq \mathbf{b}, & \text{dual:} \quad A^T \mathbf{y} \geq \mathbf{0} \\ & \mathbf{x} \geq \mathbf{0} & \mathbf{y} \geq \mathbf{0} \end{array}$$

We may note that this dual is always feasible by noting that $\mathbf{y} = \mathbf{0}$ is a feasible solution to the dual. Thus by Weak Duality, the primal is bounded. (There is another way to get to the same conclusion). Thus, by the Fundamental Theorem of Linear programming, we assert that the primal is either infeasible or has an optimal solution.

Case 1. The primal has an optimal solution \mathbf{x}^* .

Because \mathbf{x}^* is optimal to the primal, we may use Weak Duality to deduce that for every feasible \mathbf{y} to the dual i.e. satisfying $A^T \mathbf{y} \geq \mathbf{0}$ and $\mathbf{y} \geq \mathbf{0}$, we have $\mathbf{0} \cdot \mathbf{x}^* = 0 \leq \mathbf{b} \cdot \mathbf{y}$. We notice that \mathbf{x}^* is an feasible solution to the primal and satisfies $A\mathbf{x}^* \leq \mathbf{b}$ and $\mathbf{x}^* \geq \mathbf{0}$.

Case 2. The primal is infeasible.

Because we note that the Dual is feasible ($\mathbf{y} = \mathbf{0}$ is feasible), then we appeal to the Fundamental theorem of Linear programming applied to the dual to deduce that the dual either has an optimal solution or the dual is unbounded. Now the dual can't have an optimal solution \mathbf{y}^* since then by Strong Duality that would imply the primal has an optimal and hence feasible solution. Thus the dual is unbounded, which means that we can find feasible solutions smaller than any bound we wish to choose. Note that the dual is a minimization and so being unbounded means unbounded below. In particular, if we choose -1 as the bound then there will be a \mathbf{y} which is feasible ($A^T \mathbf{y} \geq \mathbf{0}$, $\mathbf{y} \geq \mathbf{0}$) with $\mathbf{b} \cdot \mathbf{y} \leq -1$ and hence $\mathbf{b} \cdot \mathbf{y} < 0$.

We can summarize this in the following result:

Theorem. Let A and \mathbf{b} be given. Either

i) there exists an \mathbf{x} with $A\mathbf{x} \leq \mathbf{b}$, $\mathbf{x} \geq \mathbf{0}$,

or

ii) there exists a \mathbf{y} with $A^T\mathbf{y} \geq \mathbf{0}$, $\mathbf{y} \geq \mathbf{0}$, $\mathbf{b} \cdot \mathbf{y} < 0$,
but not both.

Proof: We follow the above arguments. I will be explicit since I wish you to be able to produce such a proof. Our first step is to create a primal/dual pair of Linear Programs with which to make our arguments.

$$\begin{array}{rcc} \text{primal:} & \max & \mathbf{0} \cdot \mathbf{x} \\ & A\mathbf{x} & \leq \mathbf{b}, \\ & \mathbf{x} & \geq \mathbf{0} \end{array}, \quad \begin{array}{rcc} \text{dual:} & \min & \mathbf{b} \cdot \mathbf{y} \\ & A^T\mathbf{y} & \geq \mathbf{0} \\ & \mathbf{y} & \geq \mathbf{0} \end{array}.$$

We first note that the primal is always bounded. That is because the objective function always takes on the value 0. Thus, by the Fundamental Theorem of Linear programming, we assert that the primal is either infeasible or has an optimal solution.

Case 1. The primal has an optimal solution \mathbf{x}^* .

Because \mathbf{x}^* is optimal to the primal then it is also feasible and so i) holds. Using \mathbf{x}^* in Weak Duality, we deduce that for every feasible \mathbf{y} to the dual i.e. satisfying $A^T\mathbf{y} \geq \mathbf{0}$ and $\mathbf{y} \geq \mathbf{0}$, we have $\mathbf{0} \cdot \mathbf{x}^* = 0 \leq \mathbf{b} \cdot \mathbf{y}$. Thus ii) does not hold.

Case 2. The primal is infeasible.

Thus i) does not hold. We note that the Dual is feasible ($\mathbf{y} = \mathbf{0}$ is feasible) and then we appeal to the Fundamental Theorem of Linear Programming, applied to the dual, to deduce that the dual either has an optimal solution or the dual is unbounded. Now the dual can't have an optimal solution \mathbf{y}^* since then by Strong Duality that would imply the primal has an optimal and hence feasible solution. Thus the dual is unbounded, which means that we can find feasible solutions smaller than any bound we wish to choose. Note that the dual is a minimization and so being unbounded means unbounded below. (If you wish, you could translate the LP to standard inequality form to verify this). In particular, if we choose -1 as the bound then there will be a \mathbf{y} which is feasible ($A^T\mathbf{y} \geq \mathbf{0}$, $\mathbf{y} \geq \mathbf{0}$) with $\mathbf{b} \cdot \mathbf{y} < -1$ and hence $\mathbf{b} \cdot \mathbf{y} < 0$. Thus ii) holds.

In Case 1, i) holds and ii) doesn't. In Case 2, i) doesn't hold and ii) does. Since either Case 1 or Case 2 occur, we are done.

We might apply the theorem to a problem of checking if a system of inequalities $A\mathbf{x} \leq \mathbf{b}$, $\mathbf{x} \geq \mathbf{0}$ has a feasible solution. If the answer is yes, we can simply find a feasible \mathbf{x} and if the answer is no we can give a \mathbf{y} with $A^T\mathbf{y} \geq \mathbf{0}$, $\mathbf{y} \geq \mathbf{0}$, $\mathbf{b} \cdot \mathbf{y} < 0$. The \mathbf{y} gives a short proof that the system of inequalities $A\mathbf{x} \leq \mathbf{b}$, $\mathbf{x} \geq \mathbf{0}$ has no solution.

Note the interesting fact that a strict inequality has arisen when linear programming only includes inequalities \leq , \geq or equalities.