**MATH 340** A Sensitivity Analysis Example

The following examples have been sometimes given in lectures and so the fractions are rather unpleasant for testing purposes. Note that each question is imagined to be independent; the changes are not cumulative.

We wish to consider a desk manufacturer who can choose to produce three types of desks from 3 raw materials

| | Desk 1 | Desk 2 | Desk 3 | availability |
|---|---|---|---|---|
| carpentry | 4 | 6 | 8 | 600 hours |
| finishing | 1 | 3.5 | 2 | 300 hours |
| space | 2 | 4 | 3 | 550 sq. m. |
| net profit | 12 | 20 | 18 | |

Now setting $x_i$ = number of desks of type $i$ to be produced we have the LP:

$$\begin{array}{lllll} \max & 12x_1 & +20x_2 & +18x_3 & \\ & 4x_1 & +6x_2 & +8x_3 & \le 600 \\ & x_1 & +3.5x_2 & +2x_3 & \le 300 \\ & 2x_1 & +4x_2 & +3x_3 & \le 550 \end{array} \qquad x_1, x_2, x_3 \ge 0$$

We get the final dictionary:

$$\begin{array}{rcrrrr} x_1 & = & 37.5 & -2x_3 & -\frac{7}{16}x_4 & +\frac{3}{4}x_5 \\ x_2 & = & 75 & & +\frac{1}{8}x_4 & -\frac{1}{2}x_5 \\ x_6 & = & 175 & +x_3 & +\frac{3}{8}x_4 & +\frac{1}{2}x_5 \\ z & = & 1950 & -6x_3 & -\frac{11}{4}x_4 & -x_5 \end{array}$$

a) Give $B^{-1}$, appropriately labelled:

$$B^{-1} = \begin{array}{c} \\ x_1 \\ x_2 \\ x_6 \end{array} \begin{array}{c} \begin{array}{ccc} x_4 & x_5 & x_6 \end{array} \\ \begin{pmatrix} \frac{7}{16} & -\frac{3}{4} & 0 \\ -\frac{1}{8} & \frac{1}{2} & 0 \\ -\frac{3}{8} & -\frac{1}{2} & 1 \end{pmatrix} \end{array}$$

b) Give the marginal values associated with carpentry,finishing and space:

carpentry: $\frac{11}{4}$, finishing: 1, space: 0

i.e. extra hours carpentry worth $\frac{11}{4}$, extra hours finishing worth 1, extra space not helpful.

c) Give a range on $b_3$ (space) so $\{x_1, x_2, x_6\}$ still yields an optimal basis:

In this case $c_N^T - c_B^T B^{-1} A_N \le \mathbf{0}$ as before so we need $B^{-1}\mathbf{b} \ge \mathbf{0}$.

$$B^{-1}\begin{pmatrix} 600 \\ 300 \\ b_3 \end{pmatrix} = \begin{array}{c} \\ x_1 \\ x_2 \\ x_6 \end{array} \begin{array}{c} \begin{array}{ccc} x_4 & x_5 & x_6 \end{array} \\ \begin{pmatrix} \frac{7}{16} & -\frac{3}{4} & 0 \\ -\frac{1}{8} & \frac{1}{2} & 0 \\ -\frac{3}{8} & -\frac{1}{2} & 1 \end{pmatrix} \end{array} \begin{pmatrix} 600 \\ 300 \\ b_3 \end{pmatrix} = \begin{pmatrix} \frac{75}{2} \\ 75 \\ b_3 - 375 \end{pmatrix} \ge \mathbf{0}$$

Thus for $b_3 \ge 375$, we still have the same optimal basis.

d) Predict value of the optimal solution when $\mathbf{b} = (610, 310, 500)^T$:

Thus $\Delta b_1 = 10, \Delta b_2 = 10, \Delta b_3 = -50$ and so the new value of $z$ is the old value of $z$ plus $10 \times \frac{11}{4} + 10 \times 1 - 50 \times 0$ which is $1950 + \frac{75}{2} = 1987.5$. We check that

$$B^{-1}\begin{pmatrix} 610 \\ 310 \\ 500 \end{pmatrix} = \begin{pmatrix} \frac{75}{2} \\ 75 \\ 175 \end{pmatrix} + \begin{array}{c} \\ x_1 \\ x_2 \\ x_6 \end{array} \begin{array}{c} \begin{array}{ccc} x_4 & x_5 & x_6 \end{array} \\ \begin{pmatrix} \frac{7}{16} & -\frac{3}{4} & 0 \\ -\frac{1}{8} & \frac{1}{2} & 0 \\ -\frac{3}{8} & -\frac{1}{2} & 1 \end{pmatrix} \end{array} \begin{pmatrix} 10 \\ 10 \\ -50 \end{pmatrix} = \begin{pmatrix} \frac{550}{16} \\ \frac{315}{4} \\ \frac{930}{8} \end{pmatrix} \geq \mathbf{0}$$

e) Determine the range for $c_3$ so that the basis $\{x_1, x_2, x_6\}$ remains optimal:

$$c_N^T - c_B^T B^{-1} A_N = \begin{array}{c} \begin{array}{ccc} x_3 & x_4 & x_5 \end{array} \\ \begin{pmatrix} c_3 & 0 & 0 \end{pmatrix} \end{array} - \begin{array}{c} \begin{array}{ccc} x_1 & x_2 & x_6 \end{array} \\ \begin{pmatrix} 12 & 20 & 0 \end{pmatrix} \end{array} \begin{array}{c} x_1 \\ x_2 \\ x_6 \end{array} \begin{array}{c} \begin{array}{ccc} x_4 & x_5 & x_6 \end{array} \\ \begin{pmatrix} \frac{7}{16} & -\frac{3}{4} & 0 \\ -\frac{1}{8} & \frac{1}{2} & 0 \\ -\frac{3}{8} & -\frac{1}{2} & 1 \end{pmatrix} \end{array} \begin{array}{c} x_4 \\ x_5 \\ x_6 \end{array} \begin{array}{c} \begin{array}{ccc} x_3 & x_4 & x_5 \end{array} \\ \begin{pmatrix} 8 & 1 & 0 \\ 2 & 0 & 1 \\ 3 & 0 & 0 \end{pmatrix} \end{array}$$

$$= \begin{array}{c} \begin{array}{ccc} x_3 & x_4 & x_5 \end{array} \\ \begin{pmatrix} c_3 - 24 & -\frac{11}{4} & -1 \end{pmatrix} \end{array}$$

We are optimal for $c_3 \leq 24$.

A much quicker and more reasonable approach is to note the -6 as the coefficient of $x_3$ in the $z$ row and so deduce that the current $c_3$ can rise by as much as 6, i.e. $c_3 \leq 18 + 6 = 24$. Note how our sensitivity output from LINDO gives this as a reduced cost.

We can check our bound by noting that $18 \leq 24$. Note also that for $c_3 > 24$, we know that $x_3$ will be in the basis since apart from $c_3$ the problem is unchanged so if $x_3$ is not in the basis then we just have the original solution.

f) Determine the range for $c_1$ so that the basis $\{x_1, x_2, x_6\}$ remains optimal:

$$c_N^T - c_B^T B^{-1} A_N = \begin{array}{c} \begin{array}{ccc} x_3 & x_4 & x_5 \end{array} \\ \begin{pmatrix} 18 & 0 & 0 \end{pmatrix} \end{array} - \begin{array}{c} \begin{array}{ccc} x_1 & x_2 & x_6 \end{array} \\ \begin{pmatrix} c_1 & 20 & 0 \end{pmatrix} \end{array} \begin{array}{c} x_1 \\ x_2 \\ x_6 \end{array} \begin{array}{c} \begin{array}{ccc} x_4 & x_5 & x_6 \end{array} \\ \begin{pmatrix} \frac{7}{16} & -\frac{3}{4} & 0 \\ -\frac{1}{8} & \frac{1}{2} & 0 \\ -\frac{3}{8} & -\frac{1}{2} & 1 \end{pmatrix} \end{array} \begin{array}{c} x_4 \\ x_5 \\ x_6 \end{array} \begin{array}{c} \begin{array}{ccc} x_3 & x_4 & x_5 \end{array} \\ \begin{pmatrix} 8 & 1 & 0 \\ 2 & 0 & 1 \\ 3 & 0 & 0 \end{pmatrix} \end{array}$$

$$= \begin{array}{c} \begin{array}{ccc} x_3 & x_4 & x_5 \end{array} \\ \begin{pmatrix} 18 - 2c_1 & \frac{5}{2} - \frac{7}{16}c_1 & \frac{3}{4}c_1 - 10 \end{pmatrix} \end{array}$$

Thus we are optimal for $c_1 \geq 9, c_1 \geq \frac{40}{7}, c_1 \leq \frac{40}{3}$, i.e. $9 \leq c_1 \leq \frac{40}{3}$. Note $12 \in [9, \frac{40}{3}]$ which is a good check on our work.

g) Determine an optimal solution if $c_1 = 8$:

$$c_N^T - c_B^T B^{-1} A_N = \begin{array}{c} \begin{array}{ccc} x_3 & x_4 & x_5 \end{array} \\ \begin{pmatrix} 2 & -1 & -4 \end{pmatrix} \end{array}$$

Thus $x_3$ becomes an entering variable. New dictionary for basis $\{x_1, x_2, x_6\}$ is

$$\begin{array}{rclllll} x_1 & = & 37.5 & -2x_3 & -\frac{7}{16}x_4 & +\frac{3}{4}x_5 \\ x_2 & = & 75 & & +\frac{1}{8}x_4 & -\frac{1}{2}x_5 \\ x_6 & = & 175 & +x_3 & +\frac{3}{8}x_4 & +\frac{1}{2}x_5 \\ z & = & * & 2x_3 & -1x_4 & -4x_5 \end{array}$$

$x_3$ enters and $x_1$ leaves.

$$\begin{array}{rcllll} x_3 & = & \frac{75}{4} & & & \\ x_2 & = & 75 & & * & \\ x_6 & = & 175+\frac{75}{4} & & & \\ z & = & * & -x_1 & -\frac{23}{16}x_4 & -\frac{13}{4}x_5 \end{array}$$

$$\text{optimal solution: } x_3 = \frac{75}{4}, x_2 = 75, x_6 = 193.25$$

h) What is the optimal solution if $b_3 = 365$ (outside of the range given in c)). We compute

$$B^{-1}\begin{pmatrix} 600 \\ 300 \\ 365 \end{pmatrix} = \begin{pmatrix} \frac{75}{2} \\ 75 \\ -10 \end{pmatrix}$$

The final dictionary becomes:

$$\begin{array}{rcllll} x_1 & = & 37.5 & -2x_3 & -\frac{7}{16}x_4 & +\frac{3}{4}x_5 \\ x_2 & = & 75 & & +\frac{1}{8}x_4 & -\frac{1}{2}x_5 \\ x_6 & = & -10 & +x_3 & +\frac{3}{8}x_4 & +\frac{1}{2}x_5 \\ z & = & * & -6x_3 & -\frac{11}{4}x_4 & -x_5 \end{array}$$

We do a dual simplex pivot. We have $x_6$ leave. The largest $t$ such that $(\,-6 \quad -\frac{11}{4} \quad -1\,) + (\,1 \quad \frac{3}{8} \quad \frac{1}{2}\,)\,t \le \mathbf{0}$ is $t = 2$ and $x_5$ enters:

$$\begin{array}{rcllll} x_1 & = & \frac{105}{2} & & & \\ x_2 & = & 65 & & * & \\ x_5 & = & 20 & & & \\ z & = & * & -4x_3 & -2x_4 & -2x_6 \end{array}$$

$$\text{optimal solution: } x_1 = \frac{105}{2}, x_2 = 65, x_5 = 20$$

The new marginal values are carpentry: 2, finishing 0, space 2.

i) Consider a new desk with requirements of 8 hours of carpentry, 2 hours finishing, and 6 sq. m of space with a net profit of \$26 per desk. Is it profitable to produce this desk?

Let $x_7$ denote the number of desks produced of this new type. We compute

$$c_7 - c_B^T B^{-1} A_7 = 26 - (\,\tfrac{11}{4} \quad 1 \quad 0\,)\begin{pmatrix} 8 \\ 2 \\ 6 \end{pmatrix} = 2 > 0$$

Thus we will produce the new desk at optimality. Here is the final dictionary with variable $x_7$ added (we needed to compute $B^{-1}A_7$).

$$\begin{array}{rclllll} x_1 & = & 37.5 & -2x_3 & -\frac{7}{16}x_4 & +\frac{3}{4}x_5 & -2x_7 \\ x_2 & = & 75 & & +\frac{1}{8}x_4 & -\frac{1}{2}x_5 & \\ x_6 & = & 175 & +x_3 & +\frac{3}{8}x_4 & +\frac{1}{2}x_5 & -2x_7 \\ z & = & 1950 & -6x_3 & -\frac{11}{4}x_4 & -x_5 & +2x_7 \end{array}$$

We have $x_7$ enter and $x_1$ leave:

$$\begin{array}{rclllll} x_7 & = & \frac{75}{4} & & & & \\ x_2 & = & 75 & & * & & \\ x_6 & = & 137\frac{1}{2} & & & & \\ z & = & 1987\frac{1}{2} & -7x_3 & -\frac{51}{16}x_4 & -\frac{1}{4}x_5 & -\frac{1}{2}x_1 \end{array}$$

$$\text{optimal solution: } x_7 = \frac{75}{4}, x_2 = 75, x_6 = 137.5$$

j) What is the optimal solution if we add the constraint $x_1 + x_2 + x_3 \leq 100$. Think of this as a constraint on market size. Obviously the current solution is no longer feasible. We add a slack variable $x_7$ to get $x_7 = 100 - x_1 - x_2 - x_3$ and then reexpress in terms of non basic variables to get $x_7 = -\frac{25}{2} + x_3 + \frac{5}{16}x_4 - \frac{1}{4}x_5$ and get the final dictionary:

$$\begin{array}{rclllll}
x_1 & = & 37.5 & -2x_3 & -\frac{7}{16}x_4 & +\frac{3}{4}x_5 \\
x_2 & = & 75 & & +\frac{1}{8}x_4 & -\frac{1}{2}x_5 \\
x_6 & = & 175 & +x_3 & +\frac{3}{8}x_4 & +\frac{1}{2}x_5 \\
x_7 & = & -\frac{25}{2} & +x_3 & +\frac{5}{16}x_4 & -\frac{1}{4}x_5 \\
z & = & 1950 & -6x_3 & -\frac{11}{4}x_4 & -x_5
\end{array}$$

We do a dual simplex pivot. We have $x_7$ leave. The largest $t$ such that $(-6 \quad -\frac{11}{4} \quad -1) + (1 \quad \frac{5}{16} \quad -\frac{1}{4})\, t \leq \mathbf{0}$ is $t = 6$ and $x_3$ enters:

$$\begin{array}{rclllll}
x_1 & = & \frac{25}{2} & & & \\
x_2 & = & 75 & & * & \\
x_6 & = & 187\frac{1}{2} & & & \\
x_3 & = & \frac{25}{2} & 77 & & \\
z & = & * & -6x_7 & -\frac{7}{8}x_4 & -\frac{5}{2}x_5
\end{array}$$

$$\text{optimal solution: } x_1 = \frac{25}{2}, x_2 = 75, x_3 = \frac{25}{2}, x_6 = 162\frac{1}{2}$$

The new marginal values are carpentry $\frac{7}{8}$, finishing $\frac{5}{2}$, space 0, market 6.

Many other questions can be asked such as changing an entry in $A$, the matrix of the constraints. For a nonbasic variable this is reasonable (try it!). In a test environment, only one pivot suffices to get you to optimality but this is unrealistic and for some changes it may be advisable to start from scratch.