MATH 340 Practice for Quiz # 5

Consider the LP given by $A, \mathbf{b}, \mathbf{c}$, current basis, and final dictionary:

$$A = \begin{array}{c} \\ x_5 \\ x_6 \\ x_7 \end{array} \begin{array}{c} \begin{array}{ccccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 \end{array} \\ \begin{pmatrix} 1 & 3 & 1 & 0 & 1 & 0 & 0 \\ 1 & 2 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 1 \end{pmatrix} \end{array} \quad \begin{array}{c} \\ x_5 \\ x_6 \\ x_7 \end{array} \begin{array}{c} b \\ \begin{pmatrix} 4 \\ 3 \\ 2 \end{pmatrix} \end{array}$$

basis: $\{x_5, x_2, x_1\}$

$$\begin{array}{rcrrrrr} x_5 & = & 0 & -x_7 & +2x_6 & -2x_3 & +x_4 \\ x_2 & = & 1 & +x_7 & -x_6 & +x_3 & \\ x_1 & = & 1 & -2x_7 & +x_6 & -2x_3 & -x_4 \\ z & = & 8 & -x_7 & -2x_6 & & -2x_4 \end{array}$$

$$c = \begin{array}{c} \begin{array}{ccccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 \end{array} \\ \begin{pmatrix} 3 & 5 & 1 & 1 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

a) Give B^{-1}, appropriately labelled:

solution

$$B^{-1} = \begin{array}{c} \\ x_5 \\ x_2 \\ x_1 \end{array} \begin{array}{c} \begin{array}{ccc} x_5 & x_6 & x_7 \end{array} \\ \begin{pmatrix} 1 & -2 & 1 \\ 0 & 1 & -1 \\ 0 & -1 & 2 \end{pmatrix} \end{array}$$

b) Give the marginal values associated with the three raw materials:

solution

0,2,1

c) Determine an optimal solution when we replace $\mathbf{b}$ by $\begin{array}{c} \\ x_5 \\ x_6 \\ x_7 \end{array} \begin{array}{c} b \\ \begin{pmatrix} 5 \\ 3 \\ 2 \end{pmatrix} \end{array}$:

solution

$$\text{Now } B^{-1}\mathbf{b} = \begin{array}{c} \\ x_5 \\ x_2 \\ x_1 \end{array} \begin{array}{c} \begin{array}{ccc} x_5 & x_6 & x_7 \end{array} \\ \begin{pmatrix} 1 & -2 & 1 \\ 0 & 1 & -1 \\ 0 & -1 & 2 \end{pmatrix} \end{array} \begin{array}{c} \\ x_5 \\ x_6 \\ x_7 \end{array} \begin{array}{c} b \\ \begin{pmatrix} 5 \\ 3 \\ 2 \end{pmatrix} \end{array} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \geq \mathbf{0}$$

so the optimal solution is $x_5 = 1, x_2 = 1, x_1 = 1$ with the rest zero.

d) Determine an optimal solution when we replace $\mathbf{b}$ by $\begin{array}{c} \\ x_5 \\ x_6 \\ x_7 \end{array} \begin{array}{c} b \\ \begin{pmatrix} 3 \\ 3 \\ 2 \end{pmatrix} \end{array}$:

solution

$$\text{Now } B^{-1}\mathbf{b} = \begin{array}{c} \\ x_5 \\ x_2 \\ x_1 \end{array} \begin{array}{c} \begin{array}{ccc} x_5 & x_6 & x_7 \end{array} \\ \begin{pmatrix} 1 & -2 & 1 \\ 0 & 1 & -1 \\ 0 & -1 & 2 \end{pmatrix} \end{array} \begin{array}{c} \\ x_5 \\ x_6 \\ x_7 \end{array} \begin{array}{c} b \\ \begin{pmatrix} 3 \\ 3 \\ 2 \end{pmatrix} \end{array} = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$$

$$\begin{array}{rcrrrrr} x_5 & = & -1 & -x_7 & +2x_6 & -2x_3 & +x_4 \\ x_2 & = & 1 & +x_7 & -x_6 & +x_3 & \\ x_1 & = & 1 & -2x_7 & +x_6 & -2x_3 & -x_4 \\ z & = & 8 & -x_7 & -2x_6 & & -2x_4 \end{array}$$

We use the dual simplex method: x_5 leaves, $(-1,-2,0,-2)+(-1,2,-2,1)t \leq \mathbf{0}$ implies $t \leq 1$ and so x_6 enters.

$$\begin{array}{rcllllll} x_6 & = & 1/2 & & & & \\ x_2 & = & 1/2 & & * & & \\ x_1 & = & 3/2 & & & & \\ z & = & 7 & -2x_7 & -x_5 & -2x_3 & -x_4 \end{array} \qquad \text{optimal solution: } x_6 = 1/2, x_2 = 1/2, x_1 = 3/2$$

e) Determine the range for c_1 so that the basis $\{x_5, x_2, x_1\}$ remains optimal:

solution

$$c_N^T - c_B^T B^{-1} A_N = \begin{array}{c} \begin{array}{cccc} x_3 & x_4 & x_6 & x_7 \end{array} \\ \begin{pmatrix} 1 & 1 & 0 & 0 \end{pmatrix} \end{array} - \begin{array}{c} \begin{array}{ccc} x_5 & x_2 & x_1 \end{array} \\ \begin{pmatrix} 0 & 5 & c_1 \end{pmatrix} \end{array} \begin{array}{c|c} & \begin{array}{ccc} x_5 & x_6 & x_7 \end{array} \\ \begin{array}{c} x_5 \\ x_2 \\ x_1 \end{array} & \begin{pmatrix} 1 & -2 & 1 \\ 0 & 1 & -1 \\ 0 & -1 & 2 \end{pmatrix} \end{array} \begin{array}{c|c} & \begin{array}{cccc} x_3 & x_4 & x_6 & x_7 \end{array} \\ \begin{array}{c} x_5 \\ x_6 \\ x_7 \end{array} & \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix} \end{array}$$

$$= \begin{array}{c} \begin{array}{cccc} x_3 & x_4 & x_6 & x_7 \end{array} \\ \begin{pmatrix} 6-2c_1 & 1-c_1 & c_1-5 & 5-2c_1 \end{pmatrix} \end{array}$$

We are optimal for $3 \leq c_1 \leq 5$ (check: $3 \in [3,5]$).

f) Determine an optimal solution if $c_4 = 4$:

solution

$$c_4 - c_B^T B^{-1} A_4 = 4 - \begin{pmatrix} 0 & 2 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = 1 > 0$$

New dictionary for basis $\{x_5, x_2, x_1\}$ is

$$\begin{array}{rcllll} x_5 & = & 0 & -x_7 & +2x_6 & -2x_3 & +x_4 \\ x_2 & = & 1 & +x_7 & -x_6 & +x_3 & \\ x_1 & = & 1 & -2x_7 & +x_6 & -2x_3 & -x_4 \\ z & = & 8 & -x_7 & -2x_6 & & +x_4 \end{array}$$

x_4 enters and x_1 leaves.

$$\begin{array}{rcllllll} x_5 & = & 1 & & & & \\ x_2 & = & 1 & & * & & \\ x_4 & = & 1 & & & & \\ z & = & 9 & -3x_7 & -x_6 & -2x_3 & -x_1 \end{array} \qquad \text{optimal solution: } x_5 = 1, x_2 = 1, x_4 = 1$$

g) Find the optimal solution if we add a variable (product) x_8 with requirements $\begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$ and profit of 6 per unit of x_8:

solution

With current basis, coefficient of x_8 in final row is $c_8 - c_B^T B^{-1} A_8 = 6 - \begin{pmatrix} 0 & 2 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} = -1 \leq 0.$

Thus the current solution remains optimal (we won't start producing product x_8).