Proof of Marginal Values.

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We can readily obtain the marginal values interpretation for the dual variables by using the Revised Simplex Formulas. We say that y_i is the 'marginal value' or 'shadow price' for resource i as given in the ith constraint of the primal

$$a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n \leq b_i.$$

If we alter b_i by Δb_i, then we expect the objective function to change by about $y_i \Delta b_i$. This is made more precise in the following theorem. Note that the marginal values interpretation is one of the reasons that Linear Programming is so useful.

Theorem Consider the standard primal/dual pair of LPs. Let $\mathbf{x}^*$ be an optimal solution for the primal with $z^* = \mathbf{c} \cdot \mathbf{x}^*$. Let B be an optimal basis for the primal so that $\mathbf{y}^* = (\mathbf{c}_B^T B^{-1})^T = (y_1^*, y_2^*, \ldots, y_m^*)^T$ is an optimal solution to the dual. Let $\mathbf{b}' = (\Delta_1, \Delta_2, \ldots, \Delta_m)^T$ and consider the altered primal as follows:

$$\text{primal:} \quad \begin{array}{rc} \max & \mathbf{c} \cdot \mathbf{x} \\ & A\mathbf{x} \leq \mathbf{b} \\ & \mathbf{x} \geq \mathbf{0} \end{array} \qquad \text{altered primal:} \quad \begin{array}{rc} \max & \mathbf{c} \cdot \mathbf{x} \\ & A\mathbf{x} \leq \mathbf{b} + \mathbf{b}' \\ & \mathbf{x} \geq \mathbf{0} \end{array}$$

Then the optimal value of the objective function for the altered primal is $\leq z^* + (\sum_{i=1}^m y_i^* \cdot \Delta_i)$ with equality holding for $B^{-1}(\mathbf{b} + \mathbf{b}') \geq \mathbf{0}$.

Proof: We have $\mathbf{c} \cdot \mathbf{x}^* = \mathbf{c}_B \cdot \mathbf{x}_B = \mathbf{c}_B^T B^{-1}\mathbf{b}$. You might as well imagine that in fact $\mathbf{x}^*$ arises from the optimal basis B (the solution $\mathbf{x}_B^* = B^{-1}\mathbf{b}$ that aries from B is an optimal basis) but in general there can be many different optimal solutions, some not even arising from a basis/dictionary. Given that B was an optimal basis, we know the optimal value of the objective function can be written as $z^* = \mathbf{c}_B^T B^{-1}\mathbf{b}$.

As in our proof of Strong Duality, we know that $\mathbf{y} = (\mathbf{c}_B^T B^{-1})^T$ is a feasible solution to the dual and also $\mathbf{b} \cdot \mathbf{y} = \mathbf{y}^T\mathbf{b} = z^*$. As we go from the primal to the altered primal we note that $\mathbf{y}$ is still a feasible solution to the dual of the altered primal, since we have not changed A or $\mathbf{c}$. The objective function value of $\mathbf{y}$ in the dual of the altered primal is

$$(\mathbf{b} + \mathbf{b}') \cdot \mathbf{y} = \mathbf{b} \cdot \mathbf{y} + \mathbf{b}' \cdot \mathbf{y} = z^* + (\sum_{i=1}^m y_i^* \cdot \Delta_i)$$

and so by weak duality any optimal solution to the primal can have no larger objective function value.

If in addition we have $B^{-1}(\mathbf{b} + \mathbf{b}') \geq \mathbf{0}$, then we have a basic feasible solution to the altered primal $\mathbf{x}_B = B^{-1}(\mathbf{b} + \mathbf{b}')$ and the value of the objective function in the altered primal is

$\mathbf{c}_B \cdot \mathbf{x}_B = \mathbf{c}_B^T B^{-1}(\mathbf{b} + \mathbf{b}') = \mathbf{c}_B^T B^{-1}\mathbf{b} + \mathbf{c}_B^T B^{-1}\mathbf{b}' = z^* + \sum_{i=1}^m y_i^* \cdot \Delta_i$.

Thus we have feasible solutions to the altered primal and the dual of the altered primal that have the same objective function values and so by Weak Duality, we deduce that they are both optimal. ■

If there exists a non degenerate optimal basic feasible solution given by an optimal basis B and then $\mathbf{x}_B = B^{-1}\mathbf{b} > \mathbf{0}$. In matrix notation, we say $\mathbf{v} > \mathbf{0}$ if each entry of the vector $\mathbf{v}$ is greater than 0. You would immediately note that for $\mathbf{b}'$ relatively small in magnitude, you would expect that $B^{-1}(\mathbf{b} + \mathbf{b}') \geq 0$. But also the dual solution $\mathbf{y}^*$ is unique and hence $\mathbf{y}^* = (\mathbf{c}_B^T B^{-1})^T$. To see that the dual solution is unique we use complementary slackness (as we have done in the quiz 3 for example) and find that the m basic variables being strictly positive implies m equations for the values of $\mathbf{y}$. These equations correspond to the jth equation of the dual if the variable $x_j^* > 0$ and the equations $y_i = 0$ if the variable $x_{n+i}^* > 0$. If you think of this in matrix form we have the equations

$$B^T\mathbf{y} = \mathbf{c}_B$$

and hence the unique solution is $\mathbf{y}^T = \mathbf{c}_B^T B^{-1}$ as stated. If we do have degeneracy then there may be other optimal dual solutions $\mathbf{y}^*$. And then the predictive value must be diminished.

I would note that for the vector $\mathbf{b}'$ large enough and of a certain direction, it would be quite reasonable that the altered primal becomes infeasible.