Magic Coefficients

We have stated that an optimal solution to the dual has

$$y_i^* = -\text{coefficient of the } i\text{th slack of the primal.}$$

To see the origin of these coefficients, we revisit our proof of Strong Duality.

Let A be an $m \times n$ matrix. We obtain the Revised Simplex Formulas (our dictionaries!) by first writing $A\mathbf{x} \le \mathbf{b}$ as $[AI]\begin{bmatrix}\mathbf{x}\\ \mathbf{x}_S\end{bmatrix} = \mathbf{b}$ where we have n original variables (typically our decision variables) and m slack variables. Then considering the variables split into the m basic variables and the n non basic variables for some column basis B of $[AI]$ we obtain the Revised Simplex Formulas.

$$\mathbf{x}_B = B^{-1}\mathbf{b} - B^{-1}A_N\mathbf{x}_N$$

$$z = \mathbf{c}_B^T B^{-1}\mathbf{b} + (\mathbf{c}_N^T - \mathbf{c}_B^T B^{-1}A_N)\mathbf{x}_N$$

We note that $\mathbf{c}_B^T - \mathbf{c}_B^T B^{-1}B = \mathbf{0}$ and so $(\mathbf{c}_B^T - \mathbf{c}_B^T B^{-1}B)\mathbf{x}_B = 0$. This enables us to write

$$z = \mathbf{c}_B^T B^{-1}\mathbf{b} + (\mathbf{c}_N^T - \mathbf{c}_B^T B^{-1}A_N)\mathbf{x}_N + (\mathbf{c}_B^T - \mathbf{c}_B^T B^{-1}B)\mathbf{x}_B.$$

But now we have a symmetric expression for all variables, namely

$$z = \mathbf{c}_B^T B^{-1}\mathbf{b} + \sum_{i=1}^{n+m}(c_i - \mathbf{c}_B^T B^{-1}A_i)\mathbf{x}_i$$

where A_i denotes the column of $[AI]$ indexed by x_i. We regroup the variables into the original variables $\mathbf{x}$ and the slack variables $\mathbf{x}_S$ to obtain a new expression for z:

$$z = \mathbf{c}_B^T B^{-1}\mathbf{b} + (\mathbf{c}^T - \mathbf{c}_B^T B^{-1}A)\mathbf{x} + (\mathbf{0}^T - \mathbf{c}_B^T B^{-1}I)\mathbf{x}_S$$

Thus the coefficients of $\mathbf{x}_S$ in the final z-row are $\mathbf{0}^T - \mathbf{c}_B^T B^{-1}I = -\mathbf{y}^T$ when we have chosen $\mathbf{y}^T = \mathbf{c}_B^T B^{-1}$. Hence we have verified that the negatives of the coefficients of the slacks are the dual variables.