Consider the following (primal) LP

$$\text{max} \quad \mathbf{c} \cdot \mathbf{x}$$
$$A\mathbf{x} \leq \mathbf{b}\,.$$
$$\mathbf{x} \geq \mathbf{0}$$

1. **Fundamental Theorem of Linear Programming**.
   A linear program satisfies exactly one of the following:

   i) The LP is infeasible (i.e. has no feasible solution).

   ii) The LP has an optimal solution.

   iii) The LP is unbounded (i.e. (for a maximization problem) for any bound L, there exists a feasible solution $\mathbf{x}$ to the LP with $\mathbf{c} \cdot \mathbf{x} > L$)

   The following fact has been added by Chvátal:

   iv) If there is a feasible solution then there is a basic feasible solution and if there is an optimal solution then there is a basic feasible solution which is optimal.

   In comparison, iv) is little used in this course. Although the simplex algorithm only looks at basic feasible solutions, in the end we usually are not unhappy looking at any solution even if it is not a basic feasible solution. When I ask you to use the Fundamental Theorem of LP, I will be referring to i),ii),iii). The statement iv) follows from the proof of i) (Phase One) and ii) (Phase Two).

   The dual LP of the primal given above is

$$\text{min} \quad \mathbf{b} \cdot \mathbf{y}$$
$$A^T\mathbf{y} \geq \mathbf{c}\,.$$
$$\mathbf{y} \geq \mathbf{0}$$

2. **Weak Duality Theorem**. Let $\mathbf{x}$ be a feasible solution to the primal and let $\mathbf{y}$ be a feasible solution to the dual. Then

$$\mathbf{c} \cdot \mathbf{x} \leq \mathbf{b} \cdot \mathbf{y}\,.$$

If in addition we have $\mathbf{c} \cdot \mathbf{x} = \mathbf{b} \cdot \mathbf{y}$, then $\mathbf{x}$ is an optimal solution to the primal and $\mathbf{y}$ is an optimal solution to the dual.

   In the text, weak duality is given as equation (5.4). I have emphasized it because of other problems (e.g. non-linear constraints) where equality is not expected.

3. **Strong Duality Theorem**. If either
   i) the primal has an optimal solution or the dual has an optimal solution
   or
   ii) there exists feasible solutions to both the primal and the dual
   then there exists an optimal solution $\mathbf{x}^*$ to the primal and an optimal solution $\mathbf{y}^*$ to the dual with

$$\mathbf{c} \cdot \mathbf{x}^* = \mathbf{b} \cdot \mathbf{y}^*.$$

Note that Weak Duality combined with the Fundamental Theorem of Linear Programming will mean that if there exists feasible solutions to both the primal and the dual then we deduce the primal has an optimal solution.

4. **Complementary Slackness Theorem**.

Let $\mathbf{x}$ be a feasible solution to the primal and let $\mathbf{y}$ be a feasible solution to the dual. Then $\mathbf{x}$ and $\mathbf{y}$ are optimal to their respective LP's if and only if Complementary Slackness holds, namely

$$i\text{th primal slack} \cdot y_i = 0 \text{ for } i = 1, 2, \ldots, m,$$

$$x_j \cdot j\text{th dual slack} = 0 \text{ for } j = 1, 2, \ldots, n.$$

Alternatively, we can express complementary slackness as:

$$i\text{th primal slack} = 0 \text{ or } y_i = 0 \text{ or both for } i = 1, 2, \ldots, m,$$

$$x_j = 0 \text{ or } j\text{th dual slack} = 0 \text{ or both for } j = 1, 2, \ldots, n.$$

5. **Economic Interpretation of the Dual Variables**. Consider the standard primal/dual pair of LPs. Let $\mathbf{x}^*$ be an optimal solution for the primal with $z^* = \mathbf{c} \cdot \mathbf{x}^*$. Let B be an optimal basis for the primal so that $\mathbf{y}^* = (\mathbf{c}_B^T B^{-1})^T = (y_1^*, y_2^*, \ldots, y_m^*)^T$ is an optimal solution to the dual. Let $\mathbf{b}' = (\Delta_1, \Delta_2, \ldots, \Delta_m)^T$ and consider the altered primal as follows:

$$
\text{primal:} \quad
\begin{array}{c}
\max \quad \mathbf{c} \cdot \mathbf{x} \\
A\mathbf{x} \leq \mathbf{b} \\
\mathbf{x} \geq \mathbf{0}
\end{array}
\qquad
\text{altered primal:} \quad
\begin{array}{c}
\max \quad \mathbf{c} \cdot \mathbf{x} \\
A\mathbf{x} \leq \mathbf{b} + \mathbf{b}' \\
\mathbf{x} \geq \mathbf{0}
\end{array}
$$

Then the optimal value of the objective function for the altered primal is $\leq z^* + (\sum_{i=1}^m y_i^* \cdot \Delta_i)$ with equality holding for $B^{-1}(\mathbf{b} + \mathbf{b}') \geq \mathbf{0}$.

We expect $B^{-1}(\mathbf{b} + \mathbf{b}') \geq \mathbf{0}$ for small $\mathbf{b}'$. Note that for certain $\mathbf{b}'$, the altered primal may be infeasible.

If we change the right hand side of the ith inequality of the primal by 1 (i.e. replace $\mathbf{b}$ by $\mathbf{b}+1$) then we will change the optimal value of the objective function by at most y_i^* (and often by exactly y_i^*). Thus we think of $\mathbf{y}^*$ as the **marginal values** associated to the constraints of the primal. Of course, mathematically we prefer to consider changing the right hand side of the ith inequality by Δ (i.e. replace $\mathbf{b}$ by $\mathbf{b}+\Delta$) and then we will change the optimal value of the objective function by at most $y_i^* \cdot \Delta$ (with equality in many circumstances).

The dual variables are often called 'shadow prices' or 'marginal values' in view of the above result. LINDO amusingly calls them 'dual prices'. We need an economic interpretation of the LP where the objective function measures profit. Assume the variable x_j represent the amount of product j that is to be produced. The ith inequality takes the form $\sum_{j=1}^n a_{ij}x_j \leq b_i$ and so we may interpret a_{ij} as the amount of resource i required by a unit of product j and b_i is the total availability of resource i. Thus changes to b_i by Δ can be interpreted as increasing the availability of resource i by Δ. If we know the objective function changes by $y_i^* \cdot \Delta$, then the marginal value (how much profit you can make by having more of resource i) of a unit of resource i is y_i^*, the ith dual variable. That is to say changing the availability of resource i by 1 changes the objective function value by y_i^*. Of course as noted in the theorem, the marginal values may not be exactly correct. In the sensitivity section of the course we determine precise ranges for which the marginal values yield exact answers and this information is given by LINDO.