Math 340 **Some games reduce to smaller games**
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Certain payoff matrices reduce nicely and then are easy to solve. Let

$$A = \begin{bmatrix} \mathbf{r}_1^T \\ \mathbf{r}_2^T \\ \vdots \\ \mathbf{r}_m^T \end{bmatrix} = \begin{bmatrix} \mathbf{s}_1 & \mathbf{s}_2 & \cdots & \mathbf{s}_n \end{bmatrix}$$

Theorem 1 *Let A be a payoff matrix with $\mathbf{r}_i \leq \mathbf{r}_j$. Then there is an optimal strategy $\mathbf{x}^*$ with $x_i^* = 0$. Also if A is a payoff matrix with $\mathbf{s}_i \geq \mathbf{s}_j$. Then there is an optimal strategy $\mathbf{y}^*$ with $y_i^* = 0$.*

Proof: From any optimal strategy $\mathbf{x}$ for the row player, we can create a new optimal strategy $\mathbf{x}^{**}$ with $\mathbf{x}^{**} = \mathbf{x}$ except that $x_i^{**} = 0$ and $x_j^{**} = x_i + x_j$. ■

As an example, consider the following payoff matrix:

$$\begin{bmatrix} -2 & 3 & 0 & -6 & -3 \\ 0 & -4 & 9 & -2 & 1 \\ 6 & -2 & 7 & 4 & 5 \\ 7 & -3 & 8 & 3 & 2 \end{bmatrix}$$

Now column 3 is larger than column 4 and so the column player won't choose column 3. The new payoff matrix with column 3 deleted is

$$\begin{bmatrix} -2 & 3 & -6 & -3 \\ 0 & -4 & -2 & 1 \\ 6 & -2 & 4 & 5 \\ 7 & -3 & 3 & 2 \end{bmatrix}$$

Now column 1 is larger than column 3 and so the column player won't choose column 1. The new payoff matrix with column 1 deleted is

$$\begin{bmatrix} 3 & -6 & -3 \\ -4 & -2 & 1 \\ -2 & 4 & 5 \\ -3 & 3 & 2 \end{bmatrix}$$

Now row 2 and row 4 are both smaller than row 3 and so the row player won't choose row 2 or row 4. The new payoff matrix with rows 2 and 4 deleted is

$$\begin{bmatrix} 3 & -6 & -3 \\ -2 & 4 & 5 \end{bmatrix}$$

Now column 3 is larger than column 2 and so the column player won't choose column 3. The new payoff matrix with column 3 deleted is

$$\begin{bmatrix} 3 & -6 \\ -2 & 4 \end{bmatrix}$$

We compute the optimal strategy for the row player is $(2/5, 3/5)^T$ with value 0 and the optimal strategy for the column player is $(2/3, 1/3)^T$. Back in the original 4×5 game, the optimal strategies for the row and column player respectively are $\mathbf{x}^* = (2/5, 0, 3/5, 0)^T$ and $\mathbf{y}^* = (0, 2/3, 0, 1/3, 0)^T$. This problem was more for amusement than anything but it repeatedly reminds us than an optimal player does not choose a bad strategy.