Math 340 **A version of Battleship** Richard Anstee

Chapter 15 in the LP text by Chvátal has a lovely exposition concerning Two Person Zero Sum games. For those who took the course MATH 344, I apologize for an overlap. A problem can be found in the Chvátal text that can be seen as a version of battleship.

We start with a 2×3 board. One player hides a 1×2 battleship called a destroyer or patrol boat (there are 7 possible positions for the destroyer) and the other player guesses one of the 6 positions to sink the destroyer.

This is not the game of battleship which you play but it is a close relative. See the wikipedia article. In that game the various ships are of various lengths (1990 version Carrier of length 5, Battleship of length 4, cruiser or submarine of length 3, and a destroyer of length 2; 2002 version Carrier of length 5, Battleship of length 4, destroyer or submarine of length 3, and a patrol boat of length 2). In particular the larger ships in the game are easier to discover and so one could ignore them in considering a strategy. The bigger challenge is finding the small 1×2 ships of which this small game can be viewed as a version.

We obtain a payoff matrix where player 1 or the row player chooses one of 7 locations for the ship and player 2 chooses one of 6 positions to blow up a bomb.

Thus player 1 has 7 strategies for placement of the 1×2 destroyer.

1

X	X	

2 :

X	X	

3 :

	X	X

4 :

	X	X

5 :

X		
X		

6 :

	X	
	X	

7 :

		X
		X

Then player 2 has 6 strategies or choices for where to set the bomb

1	2	3
4	5	6

Now the payoff matrix $A = (a_{ij})$ is formed as the payoff to player 1 (row player) as follows

a_{ij} = payoff to player 1 if player 1(row player) plays strategy i and player 2 (column player) plays strategy j.

Note that $-a_{ij}$ would be the payoff to player 2 (column player) and hence the game is called zero sum. For our game the payoff matrix A is a 7×6 matrix as follows:

$$A = \begin{bmatrix} -1 & -1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & -1 & -1 & 1 \\ 1 & -1 & -1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 \\ -1 & 1 & 1 & -1 & 1 & 1 \\ 1 & -1 & 1 & 1 & -1 & 1 \\ 1 & 1 & -1 & 1 & 1 & -1 \end{bmatrix}$$

Such games with a payoff matrix are called *two person zero sum games* on the grounds that the payoff to one player is the negative of the payoff to the other player and so the sum of the payoffs is 0. Most activities we choose, even with two participants, have positive sum! Futures markets are negative sum games since they begin as a zero sum game but commissions reduce the payoff. Investments are typically envisioned as positive sum games; think stock dividends.

Can we give the optimal strategies for the row player and for the column player. It seems that you ought to be able to engineer an average payoff of 1/3 (for player 1) since the ship covers only 2 squares of the 6 and so you might expect to win 4 out of 6 times and lose 2 out of 6 times. Then $1/3 = (4/6) \cdot 1 + (2/6) \cdot -1$. This is not a proof but a heuristic argument so we should set up the LP to solve this game.

Let us first recall a much simpler game, Rock-Paper-Scissors. The strategies are denoted R,P,S.

$$\begin{array}{cc} & \text{player 2} \\ & \begin{array}{ccc} R & P & S \end{array} \\ \text{player 1} \quad \begin{array}{c} R \\ P \\ S \end{array} & \left[\begin{array}{rrr} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{array}\right] \end{array}$$

If you play with a 6 year old, then you might employ the following strategy. You guess the 6 year old will start with a rock so you start with paper. Then on the second round, you guess the 6 year old will try paper and so you guess scissors! Try this out. When playing with older people, this idea of guessing your opponents strategy won't work moreover it is now appropriate to consider a mixed strategy. The idea would be to choose each of the 3 strategies with probability 1/3.

Thus we now consider mixed strategies in a game with s strategies for player 1 (row player) and t strategies for player 2 (column player).

x_i = fraction of time player 1 (row player) plays strategy i so that $x_i \geq 0$

Then we form a vector corresponding to the strategy $\mathbf{x} = (x_1, x_2, \ldots, x_s)^T$ where $x_1 + x_2 + \cdots + x_s = 1$ and $\mathbf{x} \geq \mathbf{0}$

y_j = fraction of time player 2 (column player) plays strategy j

We form a vector corresponding to the strategy $\mathbf{y} = (y_1, y_2, \ldots, y_t)^T$ where $y_1 + y_2 + \cdots + y_t = 1$ and $\mathbf{y} \geq \mathbf{0}$.

We can compute the expected payoff to player 1 (which is the negative of the expected payoff to player 2) as

$$\sum_{i,j} x_i y_j a_{i,j} = \mathbf{x}^T A \mathbf{y}$$

We guessed that for Rock-Paper-Scissors the optimal strategies will be $\mathbf{x} = (1/3, 1/3.1/3)^T = \mathbf{y}$. We consider $\mathbf{x}^T A$ which is the payoff matrix for player 2 (column player) if player 1 (the row player) plays the mixed strategy $\mathbf{x}$.

$$\mathbf{x}^T A = [1/3\,1/3\,1/3] \left[\begin{array}{rrr} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{array}\right] = [0\,0\,0]$$

Thus with the row player (player 1) following the mixed strategy $\mathbf{x} = (1/3, 1/3, 1/3)^T$ the column player can do no better than 0.

Let us return to our battleship game. Your first guess might be to have the row player (player 1) use strategy $\mathbf{x} = (1/7, 1/7, \ldots, 1/7)^T$ in which case

$$[1/7\,, 1/7\,, \cdots, 1/7] \left[\begin{array}{rrrrrr} -1 & -1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & -1 & -1 & 1 \\ 1 & -1 & -1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 \\ -1 & 1 & 1 & -1 & 1 & 1 \\ 1 & -1 & 1 & 1 & -1 & 1 \\ 1 & 1 & -1 & 1 & 1 & -1 \end{array}\right] = [3/7\,, 1/7\,, 3/7\,, 3/7\,, 1/7\,, 3/7]$$

Thus if the column player (player 2) knows the mixed strategy $\mathbf{x}$ (say by playing over time) then the column player would try the mixed strategy $\mathbf{y} = [0\,,1\,,0\,,0\,,0\,,0]$ or indeed $\mathbf{y} = [0\,,1/2\,,0\,,0\,,1/2\,,0]$ with a payoff to row player of 1/7 which is less than the value 1/3 we predicted. Moreover, if the row player knows that the column player is playing $\mathbf{y} = [0\,,1/2\,,0\,,0\,,1/2\,,0]$ then the row player can adjust its strategy according to the payoff matrix $A\mathbf{y} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ -1 \\ 1 \end{bmatrix}$. So player 1 would switch to $\mathbf{x} = [0\,,0\,,0\,,0\,,1/2\,,0\,,1/2]$ and obtain expected winnings of 1 ?!?. Then the column player would adjust his strategy ad infinitum. Sounds hopeless. So we play a more conservative strategy for the players and remarkably find optimal solutions for both.

For the row player we wish to choose a mixed strategy $\mathbf{x}$ which maximizes the payoff when allowing the column player to choose any strategy $\mathbf{y}$ (even given $\mathbf{x}$).

$$\max_{\mathbf{x}} \left(\min_{\mathbf{y}} x^T A\mathbf{y} \right)$$

where we assume $\mathbf{x}$ and $\mathbf{y}$ are mixed strategies namely $\mathbf{1} \cdot \mathbf{x} = 1$, $\mathbf{x} \geq 0$ and $\mathbf{1} \cdot \mathbf{y} = 1$, $\mathbf{y} \geq \mathbf{0}$. We must show that choosing a pure strategy for $\mathbf{y}$ always makes sense in this optimization but let us delay that for the moment. Then our problem becomes

$$\max_{\mathbf{x}} \text{ minimum entry of } x^T A$$

This is a linear program although we need to recall how to make the minimum idea into linear programming constraints. We let z denote the value obtained. We wish to maximize z subject to the conditions that z is less than or equal to the jth entry of $\mathbf{x}^T A$ for each choice j. For our battleship game, the LP becomes:

$$\begin{array}{lllll}
\max & z & & & \\
 & z & -(-x_1 + x_2 + x_3 + x_4 - x_5 + x_6 + x_7) & \leq & 0 \\
 & z & -(-x_1 + x_2 - x_3 + x_4 + x_5 - x_6 + x_7) & \leq & 0 \\
 & z & -(+x_1 + x_2 - x_3 + x_4 + x_5 + x_6 - x_7) & \leq & 0 \\
 & z & -(+x_1 - x_2 + x_3 + x_4 - x_5 + x_6 + x_7) & \leq & 0 \\
 & z & -(+x_1 - x_2 + x_3 - x_4 + x_5 - x_6 + x_7) & \leq & 0 \\
 & z & -(+x_1 + x_2 + x_3 - x_4 + x_5 + x_6 - x_7) & \leq & 0 \\
 & & x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 & = & 1 \\
 & & x_i \geq 0 \text{ for } i = 1, 2, \ldots, 7 & &
\end{array}$$

In matrix notation the LP is

$$\begin{array}{lcll}
\max & z + \mathbf{0} \cdot \mathbf{x} & & \\
 & \left|\begin{array}{c|c} \mathbf{1} & -A^T \\ \hline 0 & \mathbf{1}^T \end{array}\right| \begin{bmatrix} z \\ \mathbf{x} \end{bmatrix} & \begin{array}{c} \leq \\ = \end{array} & \begin{array}{c} \mathbf{0} \\ 1 \end{array} \\
 & z \text{ free}, \ \mathbf{x} \geq \mathbf{0} & &
\end{array}$$

I chose the game battleship because the payoff matrix is not square and certainly not skew symmetric. Note that the payoff matrix for Rock-Paper-Scissors is skew symmetric, namely $A^T = -A$.

Having $A^T = -A$ is actually quite natural in a game with both players being given the same strategies (and payoffs). But it would confuse you in the LP formulation. We will have this difficulty in the LP for Morra. Don't be fooled.

Now for our battleship game, we can run this LP and solve. One strategy for the row player is $\mathbf{x} = [0,0,0,0,1/3,1/3,1/3]^T$ with $z = 1/3$ for which $\mathbf{x}^T A = [1/3,1/3,1/3,1/3,1/3,1/3]$. Thus we have a strategy for the row player that guarantees a payoff of 1/3 *even if the column player knows the strategy!* Any choice for y can only increase the payoff to row player or leave it at 1/3, which was our guess. Are there any other such strategies? If we solve the LP, we can't do better but may find some alternate strategies! You might try $\mathbf{x} = [1/3,1/3,0,0,0,0,1/3]^T$ with $z = 1/3$. How do we know this is optimal? By our duality theory, the dual solution should help. The dual is:

$$\begin{array}{ll} \min & w + \mathbf{0}\cdot\mathbf{y} \\ & \left|\begin{array}{c|c} \mathbf{1} & -A \\ \hline 0 & \mathbf{1}^T \end{array}\right| \begin{bmatrix} w \\ \mathbf{y} \end{bmatrix} \begin{array}{cc} \geq & \mathbf{0} \\ = & 1 \end{array} \\ & w \text{ free}, \ \mathbf{y} \geq \mathbf{0} \end{array}$$

This is in fact the LP that you would use to determine an optimal strategy for the column player. We can rewrite this (by our standard transformations) to get something familiar . . .

$$\begin{array}{ll} \max & -w + \mathbf{0}\cdot\mathbf{y} \\ & \left|\begin{array}{c|c} -\mathbf{1} & A \\ \hline 0 & \mathbf{1}^T \end{array}\right| \begin{bmatrix} w \\ \mathbf{y} \end{bmatrix} \begin{array}{cc} \leq & \mathbf{0} \\ = & 1 \end{array} \\ & w \text{ free}, \ \mathbf{y} \geq \mathbf{0} \end{array} \qquad \text{or} \qquad \begin{array}{ll} \max & -w + \mathbf{0}\cdot\mathbf{y} \\ & \left|\begin{array}{c|c} \mathbf{1} & A \\ \hline 0 & \mathbf{1}^T \end{array}\right| \begin{bmatrix} -w \\ \mathbf{y} \end{bmatrix} \begin{array}{cc} \leq & \mathbf{0} \\ = & 1 \end{array} \\ & w \text{ free}, \ \mathbf{y} \geq \mathbf{0} \end{array}$$

We have an LP corresponding to a game with payoff matrix $-A^T$ (instead of A) with objective function value $-w$ (instead of Z) which is exactly the game that the column player is playing. After the transformation we can see that we have replaced $-A^T$ by A for which the change $-A^T \to A^T$ corresponds to making the payoff to the column player be the negative of that reported in A and for which the change $A^T \to A$ corresponds to interchanging the roles of row and column players. This the dual is actually computing the optimal conservative strategy for the column player but $-w$ is the negative of the payoff to the row player. In any event we find a strategy $\mathbf{y} = [1/6,1/6,1/6,1/6,1/6,1/6]^T$ with $w = 1/3$ where

$$A\mathbf{y} = \begin{bmatrix} 1/3 \\ 1/3 \\ 1/3 \\ 1/3 \\ 1/3 \\ 1/3 \\ 1/3 \end{bmatrix}$$

Thus we have found a strategy for the column player that guarantees that the row player can win no more that 1/3. Thus 1/3 is the optimal solution for the row player and also the column player (we are using a weak duality idea). We will say for Battleship that the *value* of the game is 1/3 (we always refer to the payoff to the row player), which we denote $v(A) = 1/3$. By the way, the column player also has optimal strategy $\mathbf{y} = [0,1,0,1,0,1]^T$. No uniqueness here.