Math 340 Dual Simplex resulting in infeasibility Richard Anstee

Consider a primal

$$\begin{array}{c} \max \mathbf{c}\cdot\mathbf{x} \\ A\mathbf{x} \le \mathbf{b} \\ \mathbf{x} \ge \mathbf{0} \end{array}.$$

If we have a dictionary with all the coefficients in the z row are negative (namely $\mathbf{c}_N - \mathbf{c}_B B^{-1} A_N \le \mathbf{0}^T$ then we can call this *dual feasible* since $\mathbf{c}_B^T B^{-1}$ would be a feasible solution to the dual:

$$\begin{array}{c} \min \mathbf{b}\cdot\mathbf{y} \\ A^T\mathbf{y} \ge \mathbf{c} \\ \mathbf{y} \ge \mathbf{0} \end{array}.$$

If we start with a dictionary (for the primal) that is infeasible (namely $B^{-1}\mathbf{b} \not\ge \mathbf{0}$) which has all the coefficients in the z row being negative then we can proceed with the Dual Simplex algorithm. The following example gives one way that this could happen but you imagine that this could occur in a sensitivity analysis problem using the dual simplex.

$$\begin{array}{lrrcr} \max & -3x_1 & -x_2 & & \\ & 2x_1 & +2x_2 & \le & 1 \\ & -2x_1 & -x_2 & \le & -2 \\ & 4x_1 & +3x_2 & \le & 1 \end{array} \qquad x_1, x_2 \ge 0$$

We have our first dictionary

$$\begin{array}{rcrrr} x_3 & = & 1 & -2x_1 & -2x_2 \\ x_4 & = & -2 & +2x_1 & +x_2 \\ x_5 & = & 1 & -4x_1 & -3x_2 \\ z & = & & -3x_1 & -x_2 \end{array}$$

Rather than introduce x_0 and use our two phase method, we are able to embark directly on our dual simplex method. We choose x_4 to leave and then (in order to preserve dual feasibility) we choose x_2 as the entering variable. We obtain the following dictionary:

$$\begin{array}{rcrrr} x_3 & = & -3 & +2x_1 & -2x_4 \\ x_2 & = & 2 & -2x_1 & +x_4 \\ x_5 & = & -5 & +2x_1 & -3x_4 \\ z & = & -2 & -x_1 & -x_4 \end{array}$$

Note that we have made progress (we have a better dual solution with a smaller objective function value in the dual of -2 rather than 0). We choose x_5 to leave (greedily choosing the 'largest' negative coefficient) and then (in order to preserve dual feasibility) we choose x_1 as the entering variable. We obtain the following dictionary:

$$\begin{array}{rcrrr} x_3 & = & 2 & +x_5 & +x_4 \\ x_2 & = & -3 & -x_5 & -2x_4 \\ x_1 & = & 5/2 & +(1/2)x_5 & +(3/2)x_4 \\ z & = & -9/2 & -(1/2)x_5 & -(5/2)x_4 \end{array}$$

Again we have made progress finding a dual solution of value -9/2. We would choose x_2 to leave but we are unable to find an entering variable since $(\,-(1/2) \quad -(5/2)\,) + \lambda(\,-1 \quad -2\,) \le \mathbf{0}^T$ for all

$\lambda \geq 0$). So we guess that the dual is unbounded but how can we see this? A solution which is somewhat wishful thinking is taking the current dual solution $\mathbf{y} = (0, 5/2, 1/2)$ (obtained as $\mathbf{c}_B^T B^{-1}$ which is readily obtained as the coefficients of the slack variables. Now why not add t times the same coeficients from the row for x_2, namely $\mathbf{z} = (0, 2, 1)$ to obtain a solution $\mathbf{y} + t\mathbf{z} = (0, 5/2+2t, 1/2+t)$ with objective function value $-9/2 - 3t$ which shows the dual is unbounded. This wishful thinking works and you can verify that I have a parametric set of feasible dual solutions whose objective function, in the dual, goes to $-\infty$. Below I make explicit the reason why this works.

Now we have reached a place where we have a potential leaving variable but no entering variable. Imagine in general that we are doing the dual simplex method and we have x_k leaving. Let $[0 \quad 0 \quad \cdots \quad 0 \quad 1 \quad 0 \quad \cdots \quad 0]$ denote the $m \times 1$ vector with a 1 in the column corresponding to x_k. Thus

$$[0 \quad 0 \quad \cdots \quad 0 \quad 1 \quad 0 \quad \cdots \quad 0] B^{-1}\mathbf{b} < 0$$

since the cosntant entry must be zero in the row corresponding to x_k.

If we are unable to determine an entering variable then that is because

$$[0 \quad 0 \quad \cdots \quad 0 \quad 1 \quad 0 \quad \cdots \quad 0] B^{-1}A_N \geq \mathbf{0}$$

namely the entries in the row corresponding to x_k must all be negative and the entries in that row are the row of $-B^{-1}A_N$.

Now we do the standard trickery (as done in the proof of Strong Duality). We have

$$[0 \quad 0 \quad \cdots \quad 0 \quad 1 \quad 0 \quad \cdots \quad 0] B^{-1}B \geq \mathbf{0}^T$$

and so for any variable x_i we have

$$[0 \quad 0 \quad \cdots \quad 0 \quad 1 \quad 0 \quad \cdots \quad 0] B^{-1}A_i \geq \mathbf{0}.$$

Now regroup the variables by original variables and slack variables and we obtain

$$[0 \quad 0 \quad \cdots \quad 0 \quad 1 \quad 0 \quad \cdots \quad 0] B^{-1}A \geq \mathbf{0}^T$$

and

$$[0 \quad 0 \quad \cdots \quad 0 \quad 1 \quad 0 \quad \cdots \quad 0] B^{-1}I \geq \mathbf{0}^T$$

If we set $\mathbf{z}^T = [0 \quad 0 \quad \cdots \quad 0 \quad 1 \quad 0 \quad \cdots \quad 0] B^{-1}$ then we discover that $[0 \quad 0 \quad \cdots \quad 0 \quad 1 \quad 0 \quad \cdots \quad 0] B^{-1}A \geq \mathbf{0}$ implies $\mathbf{z}^T A \geq \mathbf{0}^T$ which is $A^T\mathbf{z} \geq \mathbf{0}$ and $[0 \quad 0 \quad \cdots \quad 0 \quad 1 \quad 0 \quad \cdots \quad 0] B^{-1}I \geq \mathbf{0}^T$ yields $\mathbf{z}^T \geq \mathbf{0}^T$ and so $\mathbf{z} \geq \mathbf{0}$.

We also have that the ith entry of $B^{-1}\mathbf{b} = [0 \quad 0 \quad \cdots \quad 0 \quad 1 \quad 0 \quad \cdots \quad 0] B^{-1}\mathbf{b} = \mathbf{z}^T\mathbf{b}$. Now the ith entry is less than zero, because that is why we are trying to do a dual simplex pivot. So $\mathbf{z}^T\mathbf{b} = \mathbf{b} \cdot \mathbf{z} < 0$. This is exactly what we need to have the dual be unbounded (towards $-\infty$). Assume $\mathbf{y}$ is a dual solution. The $\mathbf{y} + t\mathbf{z}$ is also a dual feasible solution and, since $\mathbf{b} \cdot \mathbf{z} = \mathbf{z}^T\mathbf{b} < 0$, we have $\mathbf{b} \cdot (\mathbf{y} + t\mathbf{z}) = \mathbf{b} \cdot \mathbf{y} + t\mathbf{b} \cdot \mathbf{z}$ and so $\lim_{t \to \infty} \mathbf{b} \cdot (\mathbf{y} + t\mathbf{z}) = -\infty$.