Math 340 Cones in Linear Programming. Richard Anstee 2020

The book has very little on this, so here are a few details written out. Given a set $V = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ of k vectors, we define the *cone* generated by V to be all positive linear combinations of the vectors in V, namely

$$\{a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_k\mathbf{v}_k \mid a_1, a_2, \ldots, a_k \geq 0\}.$$

We note that the positivity of the coefficients reminds us of the positivity of the variables in an LP. If we form a matrix $[\mathbf{v}_1\mathbf{v}_2\cdots\mathbf{v}_k]$, then the cone generated by V is $\{[\mathbf{v}_1\mathbf{v}_2\cdots\mathbf{v}_k]\mathbf{z} \mid \mathbf{z} \geq \mathbf{0}\}$. We see Linear Programming arising in this definition although for us the matrix $[\mathbf{v}_1\mathbf{v}_2\cdots\mathbf{v}_k]$ becomes A^T.

Consider the primal/dual pair:

$$\text{primal:}\quad \begin{array}{lll} \max & \mathbf{c}\cdot\mathbf{x} & \\ & A\mathbf{x} & \leq \mathbf{b} \\ & \mathbf{x}\ \text{ free} & \end{array} \qquad \text{dual:}\quad \begin{array}{lll} \min & \mathbf{b}\cdot\mathbf{y} & \\ & A^T\mathbf{y} & = \mathbf{c} \\ & \mathbf{y} \geq \mathbf{0} & \end{array}$$

The main result is the following:

Theorem 1 *Assume* $\mathbf{x}$ *is a feasible solution to the primal. Then* $\mathbf{x}$ *is an optimal solution to the primal if and only if the gradient of the objective function (namely* $\mathbf{c}$*) is contained in the cone generated by the gradients of the active constraints at* $\mathbf{x}$.

We have a few more terms to define.
The *gradient* of a linear function $a_1x_1 + a_2x_2 + \cdots + a_nx_n$ is the vector $(a_1, a_2, \ldots, a_n)^T$. For the objective function we get $\mathbf{c}$. For the ith constraint we get the ith row of A.
An *active* constraint at $\mathbf{x}$ is one with zero slack, namely it is satisfied with equality.
An *non-active* constraint at $\mathbf{x}$ is one with non zero slack, namely it is satisfied with strict inequality.

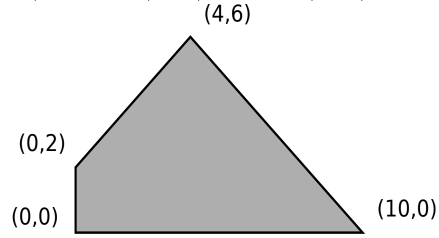
Proof: The proof of the theorem now readily follows from our duality theorems. If $\mathbf{x}$ is optimal then, by Strong Duality, there is an optimal dual solution $\mathbf{y}$ hence $A^T\mathbf{y} = \mathbf{c}$, $\mathbf{y} \geq \mathbf{0}$. Thus $\mathbf{c}$ is a positive linear combination of the gradients of the constraints. Then, using the Theorem of Complementary Slackness, we deduce that y_i is zero for the constraints are not active. Thus $\mathbf{c}$ is a positive linear combination of the gradients of the active constraints, i.e. $\mathbf{c}$ is in the cone generated by the gradients of the active constraints. ∎

There is a text book that starts the discussion of Linear Programming with this theorem which is well motivated (proven) on a geometric basis. Much like Lagrange Multipliers.

Our example considered the linear system

$$\begin{array}{lrrcr} \max & 2x_1 & +x_2 & & \\ & -x_1 & +x_2 & \leq & 2 \\ & x_1 & +x_2 & \leq & 10 \\ & -x_1 & & \leq & 0 \\ & & -x_2 & \leq & 0 \end{array}$$

There are four 'corners', $P_1 = (0,0)$, $P_2 = (0,2)$, $P_3 = (4,6)$, $P_4 = (10,0)$.

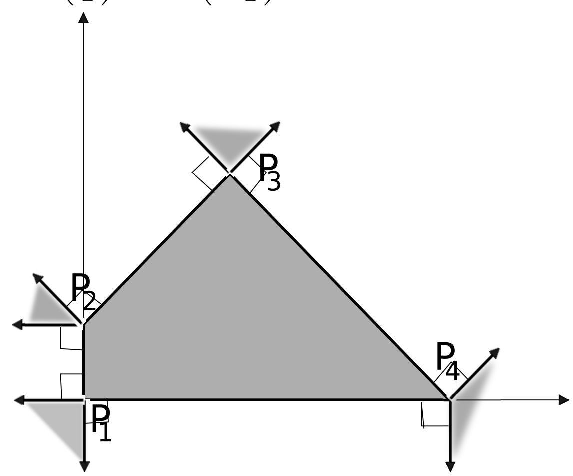


The active constraints at P_1 are the third and fourth constraints and hence the cone generated by the active constraints is $\{a_1\begin{pmatrix}-1\\0\end{pmatrix}+a_2\begin{pmatrix}0\\-1\end{pmatrix} \mid a_1,a_2\geq 0\}$.

The active constraints at P_2 are the first and third constraints and hence the cone generated by the active constraints is $\{a_1\begin{pmatrix}-1\\1\end{pmatrix}+a_2\begin{pmatrix}-1\\0\end{pmatrix} \mid a_1,a_2\geq 0\}$.

The active constraints at P_3 are the first and second constraints and hence the cone generated by the active constraints is $\{a_1\begin{pmatrix}-1\\1\end{pmatrix}+a_2\begin{pmatrix}1\\1\end{pmatrix} \mid a_1,a_2\geq 0\}$.

The active constraints at P_4 are the second and fourth constraints and hence the cone generated by the active constraints is $\{a_1\begin{pmatrix}1\\1\end{pmatrix}+a_2\begin{pmatrix}0\\-1\end{pmatrix} \mid a_1,a_2\geq 0\}$.



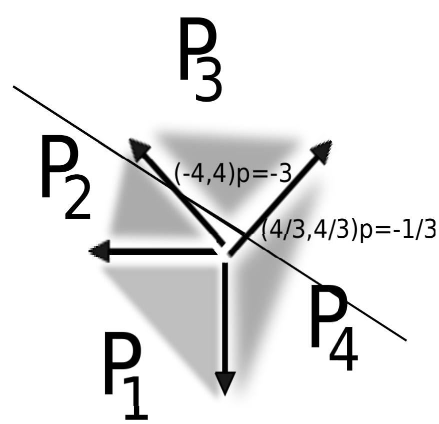
In our case $\mathbf{c}=\begin{pmatrix}2\\1\end{pmatrix}$ and we find that $\begin{pmatrix}2\\1\end{pmatrix}=2\cdot\begin{pmatrix}1\\1\end{pmatrix}+1\cdot\begin{pmatrix}0\\-1\end{pmatrix}$ and so $\mathbf{c}$ is in the cone generated by the active constraints at P_4. We deduce that P_4 is optimal without any further work.

We do not propose this as good algorithm for finding an optimal solution although the 'active set approach' does yield effective algorithms for linear and non linear programming.

The idea we wish to explore is parametric programming when for example the objective function is given as a function of a parameter p:

$$\mathbf{c}=\begin{pmatrix}2+2p\\1-p\end{pmatrix}=\begin{pmatrix}2\\1\end{pmatrix}+p\cdot\begin{pmatrix}2\\-1\end{pmatrix}$$

The 4 cones all fit together, at their tips, to form all of $\mathbf{R}^2$. The following picture may be helpful.



You might consider how this would work for a general (convex) polygon arising from inequalities particularly in 2 or 3 variables where we can draw pictures. We discover that $\mathbf{c}$ is in the cone for

P_2 for $p \leq -3$, and $\mathbf{c}$ is in the cone for P_3 for $-3 \leq p \leq -1/3$, and $\mathbf{c}$ is in the cone for P_4 for $-1/3 \leq p$. Thus, for example, when $p \leq -3$, the optimal solution is $x_1 = 0, x_2 = 2$ and so the value of the objective function is $\mathbf{c} \cdot \mathbf{x} = \begin{pmatrix} 2+2p \\ 1-p \end{pmatrix} \begin{pmatrix} 0 \\ 2 \end{pmatrix} = 2 - 2p$. Our complete parametric answer is:

$$z = \begin{cases} 2-2p & \text{for} \quad p \leq -3 \\ 14+2p & \text{for} \quad -3 \leq p \leq -1/3 \\ 20+20p & \text{for} \quad -1/3 \leq p \end{cases}$$

Note the form of the answer. The objective function z will be a piecewise linear concave continous function of p.

One observation is that the angle formed by a cone at a corner point P plus $90^{circ} + 90^\circ +$ interior angle at P sums to 360°. But the sume of the angles at corner points P is seen to sum to 360°. Imagine our feasible region is a convex polygon of n vertices. Then the sum of the interior angles at the n vertices is $180n - 360$, a formula you may have seen elsewhere. These ideas generalize. The three dimensional version is easy to visualize but a little harder to quantify what angles are.