On the economic signification of dual variables: an example

A university[1] wants to allocate its resources choosing how many slots to open in each major.

Major	Engineering	Math	Computer science
Tuition fees	4	1	2
Teaching time required	2	1	3
Space required	7	3	2

The university has 21 units of “Teaching time” and 12 units of “space”.

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- x_1 number of engineering students,
- x_2 number of math students,
- x_3 number of computer science students.

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If the university wants to maximize its revenue, it solves

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\text{maximize} & 4x_1 & +x_2 & +2x_3 & & \\
\text{s.t.} & 2x_1 & +x_2 & +3x_3 & \leqslant & 21 \\
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\end{array} \qquad x_1, x_2, x_3 \geqslant 0.$$

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With x_4, x_5 the slack variables for respectively the first and second constraints and only one pivot (x_3 enters, x_5 leaves) we get the optimal dictionary:

$$
\begin{array}{rcrrrr}
x_4 & = & 3 & +{}^{17}/_2\, x_1 & +{}^{7}/_2 x_2 & +{}^{3}/_2\, x_5 \\
x_3 & = & 6 & -{}^{7}/_2\, x_1 & -{}^{3}/_2\, x_2 & -{}^{1}/_2\, x_5 \\
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Optimality is reached and the only solution is $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 6 \end{pmatrix}$, only computer scientists!

Now the dual problem:

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\begin{array}{lcccc}
\text{minimize} & 21y_1 & +12y_2 & & \\
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As we have the final dictionary, we use it: an optimal dual solution is minus the coefficient in front of the slack variables. If some slack variables are basic, the associate dual value is 0. So here $\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

Economic interpretation:

- $y_2 = 1$. If the university increases its available space by 1 unit, its revenue goes up by 1 unit (at least for small increase). **So the university should buy more office space if the price is less than 1 unit of space per unit of money**.

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- $y_2 = 1$. If the university increases its available space by 1 unit, its revenue goes up by 1 unit (at least for small increase). **So the university should buy more office space if the price is less than 1 unit of space per unit of money**.
- $y_1 = 0$. If the university increases its available teaching time by 1 unit, its revenue does not change (at least for small increase). So the university has no incentive to increase its available teaching time. Actually with the optimal primal solution not all the available teaching time is used so it’s normal that increasing it doesn’t change anything.

An economic formulation of the dual

An subcontractor sells to the university units of teaching time and available space and wants to choose its prices.

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For the subcontractor to be profitable, considering engineering students there must hold

$$\underbrace{2y_1 + 7y_2}_{\substack{\text{Money paid by the university} \\ \text{to the subcontracor to form a student}}} \geqslant \underbrace{4}_{\substack{\text{Money the subcontractor gets} \\ \text{by forming itself the student}}}$$

Repeating this for math and computer science students, we end up with

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By strong duality, the university pays the subcontractor the same price that it will make as a revenue from tuition fees. In fact, neither the university nor the subcontractor end up making profit[2].

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