THE UNIVERSITY OF BRITISH COLUMBIA

Sessional Examination - December 18, 2010

MATH 340: Linear Programming

Instructor: Dr. R. Anstee, Section 101

Special Instructions: No calculators. You must show your work and **explain** your answers. Quote names of theorems used as appropriate. Time: 3 hours Total marks: 100

1. [12 marks]

a) [10pts] Solve the following linear programming problem, using our standard two phase method and using Anstee's rule.

$$\begin{array}{rrrrl}
\text{Maximize} & 5x_1 & +x_2 & -x_3 & \\
& 3x_1 & +x_2 & -x_3 & \leq -2 \\
& x_1 & -x_2 & -2x_3 & \leq -3 \\
& x_1 & & & \leq 2
\end{array} \qquad x_1, x_2, x_3 \geq 0$$

b) [2 marks] Give an optimal dual solution. How can you verify it is optimal?

2. [12 marks] Consider the following linear program:

$$\begin{array}{rrrrl}
\text{Maximize} & 10x_1 & +14x_2 & +20x_3 & \\
& 3x_1 & +x_2 & +3x_3 & \leq 45 \\
& 2x_1 & +2x_2 & & \leq 20 \\
& x_1 & +2x_2 & +5x_3 & \leq 15
\end{array} \qquad x_1, x_2, x_3 \geq 0$$

a) [2 marks] Give the Dual Linear Program of the above Linear Program.

b) [2 marks] State the Theorem of Complementary Slackness including a description of the conditions of complementary slackness.

c) [6 marks] You are given that an optimal primal solution has $x_1^* = 10$, $x_2^* = 0$, $x_3^* = 1$. Determine an optimal dual solution (without pivoting), stating which theorems you have used.

d) [2 marks] Does the dual solution determined in c) remain optimal if we replace the first two inequalities of the primal by $3x_1 - x_2 + 3x_3 \leq 45$ and $2x_1 + 3x_2 \leq 20$? Explain.

3. [8 marks] Given $A, \mathbf{b}, \mathbf{c}$, current basis (and B^{-1} for your computational ease), use our Revised Simplex method and Anstee's rule to determine the next entering variable (if there is one), the next leaving variable (if there is one), and the new basic feasible solution after the pivot (if there is both an entering and leaving variable). The current basis is $\{x_7, x_3, x_4\}$.

$$\begin{array}{c} \\ x_5 \\ x_6 \\ x_7 \end{array}
\begin{array}{c} \begin{array}{ccccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 \end{array} \\
\begin{pmatrix} 1 & 3 & 1 & 1 & 1 & 0 & 0 \\ -1 & 4 & 1 & 2 & 0 & 1 & 0 \\ -1 & -1 & 0 & -1 & 0 & 0 & 1 \end{pmatrix} \end{array}
\begin{array}{c} \\ x_5 \\ x_6 \\ x_7 \end{array}
\begin{array}{c} \mathbf{b} \\ \begin{pmatrix} 5 \\ 7 \\ -1 \end{pmatrix} \end{array}
\qquad B^{-1} = \begin{array}{c} \\ x_7 \\ x_3 \\ x_4 \end{array}
\begin{array}{c} \begin{array}{ccc} x_5 & x_6 & x_7 \end{array} \\
\begin{pmatrix} -1 & 1 & 1 \\ 2 & -1 & 0 \\ -1 & 1 & 0 \end{pmatrix} \end{array}$$

$$\mathbf{c} \begin{array}{c} \begin{array}{ccccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 \end{array} \\
\begin{pmatrix} 1 & 9 & 2 & 3 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

4. [25 marks] A manufacturer wishing to maximize profit can make three possible house types made from the three available resources according to the following table. We are not concerned with integrality in this question and allow fractional houses (we can think of our answer as an optimal product mix not a specific set of houses)

	house 1	house 2	house 3	total available
wood	1	2	1	6
labour	2	5	3	15
capital	3	6	4	19
$ profit	4	9	5	

Let x_i denote the amount of house i to produce and let x_{3+i} denote the ith slack for $i = 1, 2, 3$. The final dictionary is:

$$\begin{array}{rcrrrr} x_1 & = & 1 & -2x_4 & +2x_5 & -x_6 \\ x_2 & = & 2 & -x_4 & -x_5 & +x_6 \\ x_3 & = & 1 & +3x_4 & & -x_6 \\ z & = & 27 & -2x_4 & -x_5 & \end{array} \qquad B^{-1} = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \end{array} \begin{array}{c} \begin{array}{ccc} x_4 & x_5 & x_6 \end{array} \\ \begin{pmatrix} 2 & -2 & 1 \\ 1 & 1 & -1 \\ -3 & 0 & 1 \end{pmatrix} \end{array}$$

NOTE: All questions are independent of one another.

a) [2 marks] Give the marginal values for each of the resources wood, labour and capital.

b) [3 marks] Consider a new house (say house 4) with requirements 2 units of wood, 2 units of labour and 3 units of capital and profit of $7. Are you interested in producing this new house. Explain.

c) [5 marks] Give the range on c_3 (profit for house 3) so that the current solution remains optimal.

d) [5 marks] Give the range on b_2 (resource availability for labour) so that the current basis remains optimal. Also give the profit as a linear function of b_2 in that range.

Hint for e),f): You need not complete all of the very final dictionary, merely the variables in the basis and the constants and **all** the entries in the z row.

e) [5 marks] Given resource availabilities of $\begin{pmatrix} 1 \\ 4 \\ 4 \end{pmatrix}$, obtain (using the Dual Simplex method) the new optimal solution as well as the new marginal values.

f) [5 marks] Consider adding a new constraint $x_1 + x_2 \leq 2$ to our original problem. Solve using the Dual Simplex method. Report the new solution as well as the new marginal values.

5. [13 marks] We are asked to choose among various carbon reduction strategies to maximize the tons of atmospheric carbon removed subject to various constraints.

	strategy 1	strategy 2	strategy 3	maximum
capital	100	125	150	1000
labour	50	100	110	600
space	2	3	4	23
habitat damage	3	1	1	8
tons carbon removed	15	21	26	

There is an additional political decision that we must choose at least 1.5 units of Strategy 1. Here is the problem formulation for LINDO.

```
MAX     15 STRAT1 + 21 STRAT2 + 26 STRAT3
  SUBJECT TO
   CAPITAL)   100 STRAT1 + 125 STRAT2 + 150 STRAT3 <=   1000
    LABOUR)   50 STRAT1 + 100 STRAT2 + 110 STRAT3 <=   600
     SPACE)   2 STRAT1 + 3 STRAT2 + 4 STRAT3 <=   23
   HABITAT)   3 STRAT1 + STRAT2 + STRAT3 <=   8
  POLITICS)   STRAT1 >=   1.5
  END
```

Each question below is independent of each other. The LINDO input/output on this page and the next page will be useful.

a) [2 marks] If tons of carbon removed per unit of strategy 1 is reduced from 15 to 14 what is now the total tons of carbon removed by an optimal strategy?

b) [3 marks] We discover that the habitat damage is less crucial because new habitat for the endangered species has been discovered on Pender Island. What is the benefit of allowing the habitat damage to increase to 8.5 units from 8 units?

c) [3 marks] We are told that strategy 1 costs \$5000 per unit, strategy 2 costs \$6000 per unit and strategy 3 costs \$10000 per unit. Is the cost in \$ per ton of carbon removed less than \$400 per ton for our optimal strategy?

d) [2 marks] You are told of a new strategy of carbon removal using fertilizer to stimulate algae growth that will require (per unit of new strategy) 100 units of capital, 100 units of labour, 2 units space and 1 unit cost to habitat that will remove 19 tons of carbon. Should you investigate further? Explain.

e) [3 marks] A change in government has resulted in a new policy that we must choose 2 units of strategy 1. How does this affect the number of tons of carbon removed?

The following is the output from LINDO:

```
OBJECTIVE FUNCTION VALUE

       1)      113.5000

  VARIABLE        VALUE          REDUCED COST
    STRAT1         1.500000          0.000000
    STRAT2         0.000000          5.000000
    STRAT3         3.500000          0.000000


       ROW   SLACK OR SURPLUS     DUAL PRICES
  CAPITAL)       325.000000          0.000000
   LABOUR)       140.000000          0.000000
    SPACE)         6.000000          0.000000
  HABITAT)         0.000000         26.000000
 POLITICS)         0.000000        -63.000000

 NO. ITERATIONS=       0


 RANGES IN WHICH THE BASIS IS UNCHANGED:

                           OBJ COEFFICIENT RANGES
 VARIABLE         CURRENT        ALLOWABLE        ALLOWABLE
                   COEF          INCREASE         DECREASE
   STRAT1       15.000000        63.000000         INFINITY
   STRAT2       21.000000         5.000000         INFINITY
   STRAT3       26.000000         INFINITY         5.000000

                           RIGHTHAND SIDE RANGES
      ROW         CURRENT        ALLOWABLE        ALLOWABLE
                    RHS          INCREASE         DECREASE
  CAPITAL     1000.000000         INFINITY       325.000000
   LABOUR      600.000000         INFINITY       140.000000
    SPACE       23.000000         INFINITY         6.000000
  HABITAT        8.000000         1.272727         3.500000
 POLITICS        1.500000         1.166667         0.500000
```

6. [5 marks] Consider a two person zero sum game whose payoff matrix for player 1 (the 'row' player) is

$$A = \begin{pmatrix} 5 & 3 & 1 \\ 7 & 9 & 12 \end{pmatrix}$$

Give explicitly the LP that determines the optimal strategy for player 2 (the 'column' player). Do not solve. What does your objective function measure?

7. [10 marks] Let A be an $m \times n$ matrix, C be an $p \times n$ matrix, $\mathbf{b}$ be an $m \times 1$ vector and $\mathbf{d}$ be a $p \times 1$ vector. Show that either

i) There exists an $\mathbf{x}$ with $A\mathbf{x} \leq \mathbf{b}$, $C\mathbf{x} = \mathbf{d}$

or

ii) There exists $\mathbf{y}, \mathbf{z}$ with $A^T\mathbf{y} + C^T\mathbf{z} = \mathbf{0}$, $\mathbf{y} \geq \mathbf{0}$ and $\mathbf{b} \cdot \mathbf{y} + \mathbf{d} \cdot \mathbf{z} < 0$

but not both.

Name theorems used as you use them.

8. [10] Let $A, \mathbf{b}$ be given. Consider the following LP with $\mathbf{c}$ not yet specified:

$$LP(\mathbf{c}): \qquad \begin{array}{rc} \max & \mathbf{c} \cdot \mathbf{x} \\ & A\mathbf{x} \leq \mathbf{b} \\ & \mathbf{x} \geq \mathbf{0} \end{array}$$

Assume that when $\mathbf{c} = \mathbf{d}$ we have that B yields an optimal basis and that $\mathbf{x}^*$ is the optimal solution for LP($\mathbf{d}$) in this case.

a) [2 marks] Show that $\mathbf{x}^*$ is still an optimal solution for LP($1.02 \times \mathbf{d}$) namely when $\mathbf{c} = 1.02 \times \mathbf{d}$ (this might correspond to an inflation of 2%).

b) [8 marks] Let $\mathbf{e}, \mathbf{f}$ be two vectors such that B is still an optimal basis when $\mathbf{c} = \mathbf{e}$ and also when $\mathbf{c} = \mathbf{f}$. Show that $\mathbf{x}^*$ is an optimal solution for LP($2 \times \mathbf{e} + 3 \times \mathbf{f}$), namely when $\mathbf{c} = 2 \times \mathbf{e} + 3 \times \mathbf{f}$.

9. [5 marks] Consider a 'battleship' type game played on a 4×4 board. Player 1 secretly chooses a location for a domino (there are 24 possibilities but that is not so crucial to answering this question). Player 2 secretly chooses a position (among the 16 different possibilities). Player 1 wins \$1 (and player 2 loses \$1) if the domino does not occupy a position chosen by player 2 else player 2 wins \$1 (and player 1 loses \$1). One would quess that the value of the game is 12/16. Give a proof of this fact. Explicitly considering the 24×16 payoff matrix would probably be unproductive but you can use properties of the payoff matrix.