MATH 223. An averaging process. First part. Richard Anstee

The following example was given in the first lecture of a course to motivate students. I have thoughtfully waited until the fourth lecture. When we have analyzed it more, I'll give you the journal reference.

Imagine we are given n points $P_1 = (x_1, y_1)$, $P_2 = (x_2, y_2)$, ..., $P_n = (x_n, y_n)$. One could for example choose points at random. Now consider an averaging process where we relace the ith point by the average of P_i and P_{i+1} where indices are taken modulo n so that we replace the nth point by the average of P_n and P_1. Repeat this process many times. Of course the points converge to the centroid of the n points, but if we shift the points so that the centroid is the origin and if we rescale (multiply by an appropriate factor at each averaging process) we get an amazing picture. The n points are now arranged well spaced on an ellipse.

We have

$$\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i = 0 \text{ or just } \sum_{i=1}^{n} P_i = \mathbf{0}$$

This will be preserved by the averaging operation. Consider $n = 5$ and let

$$\mathrm{Avg}_5 = \begin{bmatrix} 1/2 & 1/2 & 0 & 0 & 0 \\ 0 & 1/2 & 1/2 & 0 & 0 \\ 0 & 0 & 1/2 & 1/2 & 0 \\ 0 & 0 & 0 & 1/2 & 1/2 \\ 1/2 & 0 & 0 & 0 & 1/2 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Now we see that

$$\mathrm{Avg}_5 \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} (x_1 + x_2)/2 \\ (x_2 + x_3)/2 \\ (x_3 + x_4)/2 \\ (x_4 + x_5)/2 \\ (x_5 + x_1)/2 \end{bmatrix}$$

It is often more convenient to use vector notation so that with

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}, \qquad \mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{bmatrix},$$

the new x coordinates after averaging are $\mathrm{Avg}_5\mathbf{x}$.

We can certainly deal with the x coordinates separately from the y coordinates.

To do the averaging process twice, we compute the new x and y coordinates as $\mathrm{Avg}_5(\mathrm{Avg}_5\mathbf{x}) = \mathrm{Avg}_5^2\mathbf{x}$ and $\mathrm{Avg}_5(\mathrm{Avg}_5\mathbf{y}) = \mathrm{Avg}_5^2\mathbf{y}$. Now keep averaging and it becomes clear that we need to understand Avg_5^n as $n \to \infty$.