MATH 223 There are three 2×2 matrices. Richard Anstee

These notes came from Klaus Hoechsmann, Professor Emeritus. We use the idea of similarity that arises in diagonalization, nameley we say A is similar to B if there is an invertible matrix M with $A = MBM^{-1}$. Thus B is the just A viewed in a new coordinate system. Any 2×2 matrix is similar to one of the following three matrices

$$i) \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}, \quad ii) \begin{bmatrix} r & 1 \\ 0 & r \end{bmatrix}, \quad iii) \begin{bmatrix} r & -s \\ s & r \end{bmatrix}$$

‘dilation’ ‘shear’ ‘rotation’

Note

$$\begin{bmatrix} r & -s \\ s & r \end{bmatrix} = \frac{1}{r^2+s^2} \begin{bmatrix} \frac{r}{r^2+s^2} & \frac{-s}{r^2+s^2} \\ \frac{s}{r^2+s^2} & \frac{r}{r^2+s^2} \end{bmatrix} = \rho \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

where $\rho = \frac{1}{r^2+s^2}$ and θ is chosen so that $\cos(\theta) = \frac{r}{r^2+s^2}$ and $\sin(\theta) = \frac{s}{r^2+s^2}$.

To arrive at Case i), we simply need two eigenvectors $\mathbf{v}_1, \mathbf{v}_2$ so that $M = [\mathbf{v}_1\, \mathbf{v}_2]$ is invertible. For example if we have two different eigenvalues or a repeated root of $\det(A - \lambda I)$ where we can find two eigenvectors which are not multiples of one another.

In Cases ii) and iii), use the Cayley-Hamilton Theorem

$$A^2 - \operatorname{tr}(A)A + \det(A)I = 0,$$

namely $(A - rI)^2 = qI$ (completing the square) where $r = \frac{\operatorname{tr}(A)}{2}$ and $q = \frac{\operatorname{tr}(A)^2 - 4\det(A)}{4}$.

We have $q \leq 0$ else we are in Case i).

Let $\mathbf{y}$ be chosen so it is not an eigenvector of A (thus we are not in Case i)). Try $M = [(A - rI)\mathbf{y}\ \alpha\mathbf{y}]$, where α is simply assumed to satisfy $\alpha \neq 0$ and we will specify it later.

$$(A - rI)M = [q\mathbf{y}\ \alpha(A - rI)\mathbf{y}] = M \begin{bmatrix} 0 & \alpha \\ q/\alpha & 0 \end{bmatrix}.$$

Because $\mathbf{y}$ is not an eigenvector of A and hence not an eigenvector of $A - rI$, then M is invertible. Manipulating

$$M^{-1}AM = M^{-1}(A - rI)M + M^{-1}(rI)M = \begin{bmatrix} 0 & \alpha \\ q/\alpha & 0 \end{bmatrix} \begin{bmatrix} r & 0 \\ 0 & r \end{bmatrix} = \begin{bmatrix} r & \alpha \\ q/\alpha & r \end{bmatrix}.$$

For $q = 0$, the case of repeated roots, then the existence of $\mathbf{y}$ is crucial but we can take $\alpha = 1$ and obtain Case ii), the Shear. For $q < 0$, no real roots, then we can let $s = \sqrt{-q}$ so that $q = -s^2$ and set $\alpha = -s$. This yields Case iii).

This argument seems quite specific but there are ways to generalize and ‘classify’ all $n \times n$ matrices and obtain Jordan Canonical form.