NAME: **Student no.:**

Math 223 - Midterm 2 - Friday November 7, 2008 - six pages

Please show your work. I expect some arguments for full credit.

1. [20 marks] Consider a 4×6 matrix A

$$A = \begin{bmatrix} 0 & 1 & 2 & 1 & 1 & 1 \\ 0 & 2 & 4 & 3 & 2 & 3 \\ 0 & 2 & 4 & 3 & 3 & 5 \\ 0 & 1 & 2 & 2 & 2 & 4 \end{bmatrix}$$

There is a invertible matrix E so that

$$EA = \begin{bmatrix} 0 & 1 & 2 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

a) [3 marks] What is rank(A)?
b) [8 marks] Give a basis for the row space of A and a basis for the column space of A.
c) [4 marks] Give a basis for the null space(A).
d) [5 marks] Consider a vector $\mathbf{c} \in \mathbf{R}^4$ so that $A\mathbf{x} = \mathbf{c}$ is consistent (i.e. the system of equations has a solution). Let $A' = [\mathbf{c}|\, A]$, i.e. A' is the 4×7 matrix with $\mathbf{c}$ being the first column and the remainder being the columns of A. What is rank(A')?
2. [15 marks] For the matrix

$$A = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}$$

determine explicit matrices M, D, M^{-1} where D is a diagonal matrix, so that $A = MDM^{-1}$.
3. [20 marks]

$$\text{Let} \quad \mathbf{u}_1 = \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \quad \mathbf{u}_3 = \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}$$

$$\text{NOTE:} \quad \begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & 1 \\ -1 & 2 & 3 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & -3 & 1 \\ -1 & 6 & -2 \\ 1 & -5 & 2 \end{bmatrix}$$

Let $T : \mathbf{R}^3 \longrightarrow \mathbf{R}^3$ be the linear transformation satisfying

$$T(\mathbf{u}_1) = \mathbf{u}_2, \quad T(\mathbf{u}_2) = 2\mathbf{u}_3, \quad T(\mathbf{u}_3) = 3\mathbf{u}_1.$$

a) [5 marks] Give the matrix representation of T with respect to the basis $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$.
b) [10 marks] Give the matrix representation of T with respect to the basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ (the standard basis). Give the explicit matrix with integer entries.
c) [5 marks] Give the matrix representing T^3 with respect to the basis $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$. What is the matrix representing T^3 with respect to the standard basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$? Again, you should be able to give explicit matrices.
4 [10 marks]
a) [5 marks] Explain why rank(A) = rank(A^T).
b) [5 marks] Assume A is diagonalizable. Show that A is similar to A^T.
5 [5 marks] Show that the functions e^x, e^{2x} and $x(x-1)$ are linearly independent.
6. [10 marks] Let A be a 3×3 diagonalizable matrix with $\det(A - \lambda I) = -(\lambda - 2)(\lambda - 3)(\lambda - 4)$. What are the eigenvalues of $A - 4I$?

7 [10 marks] Let A be a nonzero $n \times n$ matrix satisfying $A^k = 0$. Show that A is not diagonalizable.
8 [10 marks] Let A be an $n \times n$ matrix. Let V be the set of all matrices which are polynomials in A (of degree at most 3):

$$V = \{a_3 A^3 + a_2 A^2 + a_1 A + a_0 I \ : \ a_3, a_2, a_1, a_0 \in \mathbf{R}\}$$

a) [4 marks] Show that V is a subspace of the vector space of all $n \times n$ matrices.
b) [6 marks] Assume that $A^2 = A + 2I$. Show that $\dim(V) \leq 2$. Can $\dim(V) = 0$?