Some Viewpoints on Matrix Multiplication Richard Anstee.

Let $A=\begin{bmatrix}a & b\\ c & d\end{bmatrix}$ and $B=\begin{bmatrix}e & f\\ g & h\end{bmatrix}$. Then

$$AB=\begin{bmatrix}a & b\\ c & d\end{bmatrix}\begin{bmatrix}e & f\\ g & h\end{bmatrix}=\begin{bmatrix}ae+bg & af+bh\\ ce+dg & cf+dh\end{bmatrix}$$

There are many ways to interpret this remarkable operation. Perhaps the most natural one being the interpretation of matrix multiplication as function composition (of linear functions) from which the associativity follows directly. But the following formulas suggest other interpretations.

$$=\left[\begin{matrix} e\begin{bmatrix}a\\ c\end{bmatrix}+g\begin{bmatrix}b\\ d\end{bmatrix} & f\begin{bmatrix}a\\ c\end{bmatrix}+h\begin{bmatrix}b\\ d\end{bmatrix}\end{matrix}\right]$$

i.e. multiplication of A on the right by a matrix corresponds to column operations on A.

$$=\left[\begin{matrix} a\begin{bmatrix}e & f\end{bmatrix}+b\begin{bmatrix}g & h\end{bmatrix}\\ c\begin{bmatrix}e & f\end{bmatrix}+d\begin{bmatrix}g & h\end{bmatrix}\end{matrix}\right]$$

i.e. multiplication of B on the left by a matrix corresponds to row operations on B.

$$=\begin{bmatrix}ae & af\\ ce & cf\end{bmatrix}+\begin{bmatrix}bg & bh\\ dg & dh\end{bmatrix}=\begin{bmatrix}a\\ c\end{bmatrix}\begin{bmatrix}e & f\end{bmatrix}+\begin{bmatrix}b\\ d\end{bmatrix}\begin{bmatrix}g & h\end{bmatrix}$$

i.e. we can express AB as a sum of matrices each of rank 1.

In Matrix Algebra, you can choose which interpretation you wish to use at a given point in an argument. This multiplicity of interpretations is part of the power of Mathematics.

What follows are the same calculations for 3×3 matrices.

Let $A=\begin{bmatrix}a & b & c\\ d & e & f\\ g & h & i\end{bmatrix}$ and $B=\begin{bmatrix}q & r & s\\ t & u & v\\ x & y & z\end{bmatrix}$. Then

$$AB=\begin{bmatrix}a & b & c\\ d & e & f\\ g & h & i\end{bmatrix}\begin{bmatrix}q & r & s\\ t & u & v\\ x & y & z\end{bmatrix}=\begin{bmatrix}aq+bt+cx & ar+bu+cy & as+bv+cz\\ dq+et+fx & dr+eu+fy & ds+ev+fz\\ gq+ht+ix & gr+hu+iy & gs+hv+iz\end{bmatrix}$$

$$AB=\left[\begin{matrix} q\begin{bmatrix}a\\ d\\ g\end{bmatrix}+t\begin{bmatrix}b\\ e\\ h\end{bmatrix}+x\begin{bmatrix}c\\ f\\ i\end{bmatrix} & r\begin{bmatrix}a\\ d\\ g\end{bmatrix}+u\begin{bmatrix}b\\ e\\ h\end{bmatrix}+y\begin{bmatrix}c\\ f\\ i\end{bmatrix} & s\begin{bmatrix}a\\ d\\ g\end{bmatrix}+v\begin{bmatrix}b\\ e\\ h\end{bmatrix}+z\begin{bmatrix}c\\ f\\ i\end{bmatrix}\end{matrix}\right]$$

i.e. multiplication of A on the right by a matrix corresponds to column operations on A.

$$AB=\left[\begin{matrix} a\begin{bmatrix}q & r & s\end{bmatrix}+b\begin{bmatrix}t & u & v\end{bmatrix}+c\begin{bmatrix}x & y & z\end{bmatrix}\\ d\begin{bmatrix}q & r & s\end{bmatrix}+e\begin{bmatrix}t & u & v\end{bmatrix}+f\begin{bmatrix}x & y & z\end{bmatrix}\\ g\begin{bmatrix}q & r & s\end{bmatrix}+h\begin{bmatrix}t & u & v\end{bmatrix}+i\begin{bmatrix}x & y & z\end{bmatrix}\end{matrix}\right]$$

i.e. multiplication of B on the left by a matrix corresponds to row operations on B.

$$AB = \begin{bmatrix} aq & ar & as \\ dq & dr & ds \\ gq & gr & gs \end{bmatrix} + \begin{bmatrix} bt & bu & bv \\ et & eu & ev \\ ht & hu & hv \end{bmatrix} + \begin{bmatrix} cx & cy & cz \\ fx & fy & fz \\ ix & iy & iz \end{bmatrix}$$

$$= \begin{bmatrix} a \\ d \\ g \end{bmatrix} \begin{bmatrix} q & r & s \end{bmatrix} + \begin{bmatrix} b \\ e \\ h \end{bmatrix} \begin{bmatrix} t & u & v \end{bmatrix} + \begin{bmatrix} c \\ f \\ i \end{bmatrix} \begin{bmatrix} x & y & z \end{bmatrix}$$

i.e. we can express AB as a sum of matrices each of rank 1.