MATH 223 Jordan Forms Richard Anstee

We know that certain matrices are no diagonalizable, even over the complex numbers **C**. In the 2×2 world we have that $\det(A - \lambda I)$ is a quadratic which by the Fundamnetal Theorem of Algebra, will factor into linear factors possibly with complex roots. The only way for A to be not diagonalizable (over **C**) is for there to be a repeated root, say $\det(A - \lambda I) = (\lambda - p)^2$ and have $\dim(\text{nullsp}(A - pI)) = \dim($ eigenspace for 2$) = 1 < 2$. This would mean $\text{rank}(A - pI) = 1$. The following is an example

$$A = \begin{bmatrix} 0 & 1 \\ -4 & 4 \end{bmatrix}$$

for which $\det(A - \lambda I) = (\lambda - 2)^2$. We note $\text{rank}(A - 2I) = 1$ and so the eigenspace for eigenvalue 2 is just 1-dimensional. So A is not diagonalizable. But there is a matrix S similar to A that is perhaps easier to manipulate so that for example we can easily compute A^n. The following works

$$M = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}, \qquad S = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$$

$$\text{with } \begin{bmatrix} 0 & 1 \\ -4 & 4 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$$

But how do we compute M? We first note that $(A - 2I)^2 = 0$ which is perhaps surprising. We already know that $A - 2I$ has 0 as its only eigenvalue. We note that $(A - 2I)^2 = 0$ using our Cayley-Hamilton Theorem ! We let the first column of M be an eigenvector $\mathbf{u}$ of eigenvalue 2. We choose the second vector $\mathbf{v}$ to be the vector such that $(A - 2I)\mathbf{v} = \mathbf{u}$. You might ask whether this is possible since $\text{rank}(A - 2I) = 1$ but since $(A - 2I)^2 = 0$ we must have that $\text{colsp}(A - 2I) = \text{span}\{\mathbf{u}\}$. Now $(A - 2I)\mathbf{v} = \mathbf{u}$ and so $A\mathbf{v} = \mathbf{u} + 2\mathbf{v}$. We now compute that $AM = MS$ as desired.

This argument works for any 2×2 matrix A with $\det(A - \lambda I) = (\lambda - p)^2$ and $\dim(nullsp(A - pI)) = 1$ (so that A is not diagonalizable) where our target similar matrix is now

$$S = \begin{bmatrix} p & 1 \\ 0 & p \end{bmatrix}$$

How does this generalize to 3×3 matrices. Imagine we have 3×3 matrix A with $\det(A - \lambda I) = -(\lambda - p)^3$ and $\dim(nullsp(A - pI)) = 1$ so that $\text{rank}(A - pI) = 2$. Perhaps A is similar to

$$S = \begin{bmatrix} p & 1 & 0 \\ 0 & p & 1 \\ 0 & 0 & p \end{bmatrix}?$$

Now let $\mathbf{u}$ be an eigenvector of eigenvalue p. We wish to choose a $\mathbf{v}$ so that $A\mathbf{v} = \mathbf{u} + p\mathbf{v}$ and a $\mathbf{w}$ with $A\mathbf{w} = \mathbf{v} + p\mathbf{w}$. If we have these three vectors and form $M = [\mathbf{u}\,\mathbf{v}\,\mathbf{w}]$, then $AM = MS$. If M is invertible, then we are done.

We would proceed as before using the Cayley-Hamilton Theorem that will state $(A - pI)^3 = 0$. If we write $(A - pI)^3 = (A - pI)(A - pI)^2 = 0$, we deduce that every column of $(A - pI)^2$ is in the eigenspace of A of eigenvalue p and so $\text{colsp}(A - pI)^2 = \text{span}(\mathbf{u})$. Of course $\dim(\text{colsp}(A - pI)) = 2$ and contains $\mathbf{u}$ since $\text{colsp}(A - pI)^2 = \text{span}(\mathbf{u})$.

Now $A\mathbf{v} = \mathbf{u} + p\mathbf{v}$ yields $(A - pI)\mathbf{v} = \mathbf{u}$ and $A\mathbf{w} = \mathbf{v} + p\mathbf{w}$ yields $(A - 2I)\mathbf{w} = \mathbf{v}$ so that $(A - pI)^2\mathbf{w} = \mathbf{u}$. Let us solve for $\mathbf{w} \neq \mathbf{0}$ with knowledge that $\text{colsp}(A - pI)^2 = \text{span}(\mathbf{u})$. Now let $\mathbf{v} = (A - pI)\mathbf{w}$ so that $(A - pI)\mathbf{v} = (A - pI)^2\mathbf{w} = \mathbf{u}$. Is it true that $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are linearly independent? Assume that $a\mathbf{u} + b\mathbf{v} + c\mathbf{w} = \mathbf{0}$. Multiplying on the left by $(A - pI)^2$, we obtain $(A-pI)^2(a\mathbf{u}+b\mathbf{v}+c\mathbf{w}) = a(A-pI)^2\mathbf{u}+b(A-pI)^2\mathbf{v}+c(A-pI)^2 = c\mathbf{u} = \mathbf{0}$ from which we deduce that $c = 0$. Now mulitplying on the left by $(A-pI)$ yields $(A-pI)(a\mathbf{u}+b\mathbf{v}) = a(A-pI)\mathbf{u}+b(A-pI)\mathbf{v} = b\mathbf{u} = \mathbf{0}$ from which we deduce that $b = 0$. We now can conclude $a = 0$ (since $\mathbf{u} \neq \mathbf{0}$). Thus $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are linearly independent. Thus M is invertible and so we have shown that A is similar to S.

These ideas generalize. Our 2×2 and 3×3 matrices S are called Jordan blocks. You can read up on Jordan canonical form. .