Imagine a vector spave V. The notion of the dot product is very productive for vectors in $\mathbf{R}^n$ depending how you interpret vectors. But we do wish to consider vectors that are naturally though of as functions etc. We axiomatize the properties of the dot product as an inner product. In what follows $\mathbf{u}, \mathbf{v} \in V$ and $k \in F$.

$$< \mathbf{u}, \mathbf{v} > \in \mathbf{R} \quad \text{i.e.} \quad < \cdot, \cdot >: V \times V \to \mathbf{R}$$

$$< \mathbf{u}, \mathbf{u} > \geq 0 \text{ and } < \mathbf{u}, \mathbf{u} >= 0 \text{ if and only if } \mathbf{u} = 0.$$

$$< \mathbf{u}, \mathbf{v} >=< \mathbf{v}, \mathbf{u} >$$

$$< k\mathbf{u}, \mathbf{v} >= k < \mathbf{u}, \mathbf{v} >$$

$$< \mathbf{u}, \mathbf{v} + \mathbf{w} >=< \mathbf{u}, \mathbf{v} > + < \mathbf{u}, \mathbf{w} >$$

Obviously the dot product is an example of an inner product. We define an inner product space as a vector space with a given inner product. But certainly the choice of inner product is not unique and often refers to the 'life a vector has beyond its role as a vector in a vector space'. The following integral definition of an inner product, when the vectors have a life as a function, has many applications

$$< f, g >= \int_a^b f(x)g(x)dx \tag{1}$$

We can imagine the angle between vectors $\mathbf{u}, \mathbf{v}$ as $\dfrac{< \mathbf{u}, \mathbf{v} >}{(\sqrt{< \mathbf{u}, \mathbf{u} >}\sqrt{< \mathbf{v}, \mathbf{v} >})}$

even when there is no direct geometric picture. We use this for deciding orthogonality:

$$\mathbf{u}, \mathbf{v} \text{ are orthogonal if } < \mathbf{u}, \mathbf{v} >= 0$$

Now consider the $\sin, \cos$ functions on $[0, 2\pi]$ as well as the constant function equal to 1: $\mathbf{1}$. We use the integral definition (1) above with $a = 0$ and $b = 2\pi$. In what follows $m, n \in \{1, 2, 3, \ldots\}$.

$$\int_0^{2\pi} \sin(nx)\sin(mx)dx = 0 \text{ and } \int_0^{2\pi} \cos(nx)\cos(mx)dx = 0 \text{ for } m \neq n$$

$$\int_0^{2\pi} \sin(nx)\cos(mx)dx = 0 \text{ for any } m, n$$

$$\int_0^{2\pi} \mathbf{1} \cdot \sin(nx)dx = 0 = \int_0^{2\pi} \mathbf{1} \cdot \cos(nx)dx$$

$$\int_0^{2\pi} \sin(nx)\sin(nx)dx = \int_0^{2\pi} \cos(nx)\cos(nx)dx = \pi, \quad \int_0^{2\pi} \mathbf{1} \cdot \mathbf{1}dx = 2\pi.$$

Thus our integrals are showing that with this interpetation with integration on $[0, 2\pi]$, the functions $\mathbf{1}, \sin(x), \cos(x), \sin(2x), \cos(2x), \sin(3x), \cos(3x) \ldots$ are orthogonal. You might now have an idea what orthogonal polynomials could be.