MATH 223: Directed Paths and Matrix Theory.

A *directed graph* $D = (N, A)$ consist of a finite set N of nodes and a set A of arcs, each arc consisting of an ordered pair of vertices. A *graph* (or *undirected* Graph) has the edges consisting of unordered pairs of vertices.

It is easy to define the adjacency matrix in te same way as for graphs.

$$B = (a_{ij}), \qquad a_{ij} = \begin{cases} 1 & \text{if } (i,j) \in A \\ 0 & \text{if } (i,j) \notin A \end{cases}$$

Such a (0,1)-matrix would not typically be symmetric. Do note that for an undirected graph $G = (V, E)$, its adjacency matrix is symmetric and can be obtained as the adjacency matrix of a directed graph obtained by replacing each edge $\{i, j\}$ by the two arcs $(i, j), (j, i)$.

The most interesting property of the adjacency matrix involves directed paths. We say $v_0 v_1 v_2 \ldots, v_k$ is a directed path if $(v_i, v_{i+1}) \in A$ for each $i = 0, 1, \ldots, k-1$. Alternatively we could give the path by the sequence of arcs $a_1, a_2, a_3, \ldots, a_k$ where $a_1 = (v_0, v_1), a_2 = (v_1, v_2), \ldots, a_k = (v_{k-1}, v_k)$. Such a directed path is said to have length k.

Theorem. The B^k has its i, j entry equal to the number of directed paths from i to j using k arcs.

Proof: The most straightforward proof is by induction on k. the result is easy for $k = 1$ and to prove the inductive step we use the equation $B^k = BB^{k-1}$ as well as the entries for B^{k-1}.

A rather strange application of this is to consider the *diameter* of a Digraph, denoted diam(D). For each pair of nodes i, j, compute the shortest length (in number of arcs) of a directed path starting at i and ending at j. The diameter of D is the maximum, over all pairs i, j, of shortest lengths. (You may verify that this definition coincides with the definition of the diameter of a circle when you define shortest path between points as the shortest distance between points). The diameter of a digraph D, can be determined as the smallest k such that $(B+I)^k > 0$, where a matrix is said to be bigger than the zero matrix if each entry is bigger than 0. Such an observation, yields the fastest algorithm for computing the diameter using fast matrix multiplication algorithms under certain conditions.

We also see that for d being the diameter of the digraph and for $t > d$,rank$(I, B, B^2, B_3, \ldots, B^t) \geq$ rank$(I, B, B^2, \ldots, B^d) = d + 1$. Here rank refers to rank in $\mathbf{R}^{n^2}$ when the matrices are viewed as vectors with n^2 entries. The latter equality is proved by noting that the matrices $I, B, B^2, \ldots, B^d$ are linearly independent as vectors in $\mathbf{R}^{n^2}$ since each subsequent B^i can be shown to have one new entry nonzero which is zero for all matrices $I, B, B^2, \ldots, B^{i-1}$ (to see this take a shortest directed path of length d, say $v_0 v_1 v_2 \ldots \mathbf{v}_d$ and deduce that $v_0 v_1 v_2 \ldots \mathbf{v}_p$ is a shortest directed path of length p from v_0 to v_p and hence the (v_0, v_p)th entry of B^p is nonzero but the (v_0, v_p)th entry of B^q is nonzero for all $q < p$).