We have a determinant function $\det : \mathbf{R}^{n\times n} \to \mathbf{R}$ that satisfies various natural properties.

- $\det I = 1$
- If $B$ is obtained by multiplying row $i$ of $A$ by $t$ then $\det(B) = t\cdot\det(A)$
- If $B$ is obtained from $A$ by interchanging row $i$ and row $j$ then $\det(B) = -\det(A)$
- If $B$ is obtained from $A$ by adding a multiple of row $i$ to row $j$ then $\det(B) = \det(A)$
- $\det(AB) = \det(A)\det(B)$
- $\det(A) \neq 0$ if and only if $A$ has an inverse if and only if there exists an $\mathbf{x} \neq \mathbf{0}$ with $A\mathbf{x} = \mathbf{0}$.
- $\det(A^T) = \det(A)$
- $\det(A)$ measures some volume: $|\det(A)|$ is the volume of the parallelepiped formed by the column vectors of $A$

The idea is to give a specific function and then verify that it has the desired properties (will take several lectures). For convenience, use the notation $M_{ij}$ to denote the matrix obtained from $A$ by deleting row $i$ and column $j$. We define

$$\det(A) = (-1)^{1+1}a_{11}\det(M_{11}) + (-1)^{1+2}a_{12}\det(M_{12}) + \cdots + (-1)^{1+n}a_{1n}\det(M_{1n}) \tag{1}$$

This is called *expansion about the first row.* One of our goals is to show that the following formulas are equivalent the first being expansion about row$i$:

$$\det(A) = (-1)^{i+1}a_{i1}\det(M_{i1}) + (-1)^{i+2}a_{i2}\det(M_{i2}) + \cdots + (-1)^{i+n}a_{in}\det(M_{in}) \tag{2}$$

and the second being expansion about the $j$th column:

$$\det(A) = (-1)^{1+j}a_{1j}\det(M_{2j}) + (-1)^{2+j}a_{2j}\det(M_{2j}) + \cdots + (-1)^{n+j}a_{nj}\det(M_{nj}) \tag{3}$$

You might check that this formula works well for $2\times 2$ matrices and is the same as our previously given formula.

Computations:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \quad M_{11} = \begin{bmatrix} . & . & . \\ . & 5 & 6 \\ . & 8 & 9 \end{bmatrix} \quad M_{12} = \begin{bmatrix} . & . & . \\ 4 & . & 6 \\ 7 & . & 9 \end{bmatrix} \quad M_{13} = \begin{bmatrix} . & . & . \\ 4 & 5 & . \\ 7 & 8 & . \end{bmatrix}$$

$$\begin{aligned} \det(A) &= (-1)^{1+1}a_{11}\det(M_{11}) + (-1)^{1+2}a_{12}\det(M_{12}) + (-1)^{1+3}a_{13}\det(M_{13}) \\ &= \det(\begin{bmatrix} 5 & 6 \\ 8 & 9 \end{bmatrix}) - 2\det(\begin{bmatrix} 4 & 6 \\ 7 & 9 \end{bmatrix}) + 3\det(\begin{bmatrix} 4 & 5 \\ 7 & 8 \end{bmatrix}) = \\ &= (1\times -3) + (-2\times -6) + (3\times -3) = 0 \end{aligned}$$

Or perhaps you prefer expansion about second column

$$M_{12} = \begin{bmatrix} . & . & . \\ 4 & . & 6 \\ 7 & . & 9 \end{bmatrix} \quad M_{22} = \begin{bmatrix} 1 & . & 3 \\ . & . & . \\ 7 & . & 9 \end{bmatrix} \quad M_{32} = \begin{bmatrix} 1 & . & 3 \\ 4 & . & 6 \\ . & . & . \end{bmatrix}$$

$$\det(A) = (-1)^{1+2}a_{12}\det(M_{12}) + (-1)^{2+2}a_{22}\det(M_{22}) + (-1)^{3+2}a_{32}\det(M_{32})$$

$$= (-2)\det(\begin{bmatrix} 4 & 6 \\ 7 & 9 \end{bmatrix}) + 5\det(\begin{bmatrix} 1 & 3 \\ 7 & 9 \end{bmatrix}) + (-8)\det(\begin{bmatrix} 1 & 3 \\ 4 & 6 \end{bmatrix}) =$$

$$= (-2 \times -6) + (5 \times -12) + (-8 \times -6) = 0$$

All that work for nothing ? ! You could practice using expansion about second row and (hopefully) obtain the same result.

You should practice an example $\det(A - \lambda I)$ to look for eigenvalues/eigenvectors. We later discover that Gaussian Elimination can help us compute determinants, but this does not work well with variables such as $\lambda$ for which inadvertent division by zero might occur.

Here is an example of a surprising formula for the inverse using determinants. For a matrix $A$, we say the $i, j$ cofactor is $(-1)^{i+j}\det(M_{ij})$.

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \quad A^{-1} = \begin{bmatrix} 0 & -1 & 1 \\ -1/2 & 1/2 & 1/2 \\ 1/2 & 1/2 & -1/2 \end{bmatrix}$$

A formula for the inverse involves determining the transpose of the matrix of cofactors

$$A^* = \begin{bmatrix} (-1)^{1+1}\det(M_{11}) & (-1)^{2+1}\det(M_{21}) & (-1)^{3+1}\det(M_{31}) \\ (-1)^{1+2}\det(M_{12}) & (-1)^{2+2}\det(M_{22}) & (-1)^{3+2}\det(M_{32}) \\ (-1)^{1+3}\det(M_{13}) & (-1)^{2+3}\det(M_{23}) & (-1)^{3+3}\det(M_{33}) \end{bmatrix}$$

$$= \begin{bmatrix} \det(\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}) & -\det(\begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix}) & \det(\begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix}) \\ -\det(\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}) & \det(\begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}) & -\det(\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}) \\ \det(\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}) & -\det(\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}) & \det(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}) \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 2 & -2 \\ 1 & -1 & -1 \\ -1 & -1 & 1 \end{bmatrix}$$

Now divide by $\det(A) = -2$ to get the inverse! I would point out that the diagonal entries of $AA^*$ are all $\det(A)$ but a little more work is needed to see the off diagonals to be 0. This is not an efficient formula but useful in understanding the inverse. This part of Cramer's Rule.