When solving a quadratic (over $\mathbf{R}$) you may find there are no roots but you notice that you get expressions involving $\sqrt{-1}$. Rather than interpret $\sqrt{-1}$ as a number, we can proceed as follows. We define

$$\mathbf{C} = \{a + bi \,:\, a, b \in \mathbf{R}\}$$

and show that with a suitable multiplication, that this is a field. Let $z = a + bi$ and $w = c + di$. Define

$$zw = (a + bi)(c + di) = (ac - bd) + (ad + bc)i$$

This is most naturally interpreted as saying $i^2 = -1$ although the formula could be viewed as an abstract operation. We could interpret elements of $\mathbf{C}$ as 2-tuples and define a multiplication of 2-tuples as

$$\begin{bmatrix} a \\ b \end{bmatrix} \cdot \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} ac - bd \\ ad + bc \end{bmatrix}$$

We then have to verify that this operation combined with the standard addition yields a field. I don't think that this is so helpful, it is easiest to think of $i = \sqrt{-1}$, but there are always different points of view.

We define the real part of z as $Re(z) = a$ and the imaginary part of z as $Im(z) = b$. Here we are using the interpretation $i = \sqrt{-1}$ which we view as imaginary. We say $z \in \mathbf{R}$ when $Im(z) = 0$ although you might say this is an abuse of notation. We have always done the same with rationals $\mathbf{Q}$ and $\mathbf{R}$ and interpret $\mathbf{Q} \subset \mathbf{R} \subset \mathbf{C}$.

Define in the natural way

$$z + w = (a + c) + (b + d)i$$

To check field axioms we need $0 = 0 + 0i$ and $1 = 1 + 0i$ and we need multiplicative inverses

$$z^{-1} = \frac{a - bi}{a^2 + b^2} = \frac{a}{a^2 + b^2} - \frac{b}{a^2 + b^2} i$$

Thus z^{-1} exists if $z \neq 0$. But doing this computaion by hand is some work. $(1/2 + 2i)^{-1} =?$

One useful operation on complex numbers is complex conjugation. Define

$$z = a + bi, \qquad \overline{z} = a - bi$$

We note that $z\overline{z} = a^2 + b^2$, which you can see is yielding the multiplicative inverses. Moreover $z\overline{z} \in \mathbf{R}$. You should also note that $z + \overline{z} \in \mathbf{R}$ which we use repeatedly. We can check that $\overline{zw} = \overline{z}\,\overline{w}$ since

$$\overline{zw} = (ac - bd) - (ad + bc)i \text{ and } \overline{z}\,\overline{w} = (a - bi)(c - di) = (ac - bd) - (ad + bc)i$$

It is somewhat simpler to note that $\overline{z + w} = \overline{z} + \overline{w}$. Then we obtain very useful formulas. Assume $A\mathbf{x} = \lambda\mathbf{x}$ where we have $A \in \mathbf{C}^{n\times n}$, $\mathbf{x} \in \mathbf{C}^n$ and $\lambda \in \mathbf{C}$. Then

$$\overline{A\mathbf{x}} = \overline{\lambda\mathbf{x}} \text{ becomes } \overline{A}\overline{\mathbf{x}} = \overline{\lambda}\,\overline{\mathbf{x}}$$

If $A \in \mathbf{R}^{n\times n}$, then $\overline{A} = A$. Thus if we have an eigenvector $\mathbf{x}$ of eigenvalue λ for A, then $\overline{\mathbf{x}}$ is an eigenvector of eigenvalue $\overline{\lambda}$ for A.

Example

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad \det(A - \lambda I) = \lambda^2 + 1 = (\lambda - i)(\lambda + i)$$

For eigenvalue $\lambda = i$ we find the eigenvector

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} -i \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ i \end{bmatrix} = \begin{bmatrix} -i \\ 1 \end{bmatrix} \cdot i$$

Our previous remark gives us an eigenvector of eigenvalue $-i$:

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} i \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -i \end{bmatrix} = \begin{bmatrix} i \\ 1 \end{bmatrix} \cdot (-i)$$

This a lovely example of two for the price of one.

Perhaps the most amazing fact is the formula for $e^z = e^{a+bi}$. We note $e^{a+bi} = e^a e^{bi}$. The expression e^a is easy since $a \in \mathbf{R}$. For e^{bi} we try our usual formula for the exponential

$$\begin{aligned} e^{bi} &= 1 + (bi) + \frac{1}{2!}(bi)^2 + \frac{1}{3!}(bi)^3 + \frac{1}{4!}(bi)^4 + \frac{1}{5!}(bi)^5 + \frac{1}{6!}(bi)^6 + \cdots \\ &= 1 - \frac{1}{2!}(b)^2 + \frac{1}{4!}(b)^4 - \frac{1}{6!}(b)^6 + \cdots \\ &\quad + i\left(b - \frac{1}{3!}(b)^3 + \frac{1}{5!}(b)^5 + \cdots\right) \\ &= \cos b + (\sin b)i. \end{aligned}$$

This is amazing in that it relates the exponential function to the sine and cosine functions which may come as quite a surprise. They are not usually spoken of together in your earlier courses.

A DE system that relates to this is the following

$$\frac{d}{dt}\frac{d}{dt}y(t) = -y(t)$$

This is a second order DE. But by introducing the derivative $y'(t) = \frac{d}{dt}y(t)$ we have

$$\frac{d}{dt}\begin{bmatrix} y(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} y(t) \\ y'(t) \end{bmatrix}$$

In analogy to our previous solutions of DE's, we obtain a general solution

$$\begin{bmatrix} y(t) \\ y'(t) \end{bmatrix} = c_1 e^{it} \begin{bmatrix} -i \\ 1 \end{bmatrix} + c_2 e^{-it} \begin{bmatrix} i \\ 1 \end{bmatrix} = c_1(\cos(t) + i\sin(t)) \begin{bmatrix} -i \\ 1 \end{bmatrix} + c_2(\cos(t) - i\sin(t)) \begin{bmatrix} i \\ 1 \end{bmatrix}$$

You should be slightly worried that we are writing what looks like complex functions for a problem which is surely restricted to reals. If we start with the initial conditions $y(0) = 1$ and $y'(0) = 0$ we can solve for c_1, c_2 and hopefully real solutions result.

$$\begin{aligned} \begin{bmatrix} y(0) \\ y'(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} &= c_1 \begin{bmatrix} -i \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} i \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} -i & i \\ 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \end{aligned}$$

So

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} -i & i \\ 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}i & \frac{1}{2} \\ -\frac{1}{2}i & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}i \\ -\frac{1}{2}i \end{bmatrix}$$

We compute

$$\begin{bmatrix} y(t) \\ y'(t) \end{bmatrix} = \frac{1}{2}i \cdot (\cos(t) + i\sin(t)) \begin{bmatrix} -i \\ 1 \end{bmatrix} + -\frac{1}{2}i \cdot (\cos(t) - i\sin(t)) \begin{bmatrix} i \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2}(\cos(t) + i\sin(t)) + \frac{1}{2}(\cos(t) - i\sin(t)) \\ \frac{1}{2}(\sin(t) + i\cos(t)) + \frac{1}{2}(-\sin(t) - i\cos(t)) \end{bmatrix} = \begin{bmatrix} \cos(t) \\ -\sin(t) \end{bmatrix}$$

which is easily checked as the solution to the differential equations and satisfies the initial conditions.

You may note that $\overline{c_1} = c_2$, $\overline{e^{it}} = e^{-it}$, and the eigenvectors are conjugates and so the result must be real!