Let V be a vector space of dimension k with two bases B, C:

$B = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$,

$C = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k\}$.

Since C is a basis, we can write

$\mathbf{v}_1 = a_{11}\mathbf{u}_1 + a_{21}\mathbf{u}_2 + \cdots$

$\mathbf{v}_2 = a_{12}\mathbf{u}_1 + a_{22}\mathbf{u}_2 + \cdots$,

etc

Letting $M = (a_{ij})$ we have obtained the change of basis matrix

$$\underset{C \leftarrow B}{M} = \begin{bmatrix} a_{11} & a_{12} & \cdots \\ a_{21} & a_{22} & \cdots \\ & \vdots & \end{bmatrix}$$

Since B is a basis we can similarly write

$\mathbf{u}_1 = b_{11}\mathbf{v}_1 + b_{21}\mathbf{v}_2 + \cdots$

$\mathbf{u}_2 = b_{12}\mathbf{v}_1 + b_{22}\mathbf{v}_2 + \cdots$, etc

etc

$$\underset{B \leftarrow C}{N} = \begin{bmatrix} b_{11} & b_{12} & \cdots \\ b_{21} & b_{22} & \cdots \\ & \vdots & \end{bmatrix}$$

We check

$\mathbf{u}_i = b_{1i}\mathbf{v}_1 + b_{2i}\mathbf{v}_2 + \cdots = b_{1i}(a_{11}\mathbf{u}_1 + a_{21}\mathbf{u}_2 + \cdots) + b_{2i}(a_{12}\mathbf{u}_1 + a_{22}\mathbf{u}_2 + \cdots) + \cdots$

Now the coefficient of $\mathbf{u}_i$ must be one since C is a basis with unique coordinates and similarly the coefficient of $\mathbf{u}_j$ for $i \neq j$ must be 0.

Thus $1 = b_{1i}a_{i1} + b_{2i}a_{i2} + \cdots = a_{i1}b_{1i} + a_{i2}b_{2i} + \cdots$ which is the ii entry of MN.

Also $0 = b_{1i}a_{j1} + b_{2i}a_{j2} + \cdots = a_{j1}b_{1i} + a_{j2}b_{2i} + \cdots$ which is the ij entry of MN for $i \neq j$. We have shown $MN = I$ and so M is invertible.