MATH 223: Birds and Eigenvalues and Eigenvectors. Richard Anstee

An application to bird populations (Leslie Matrix).

Sample computation

Let

$$A = \begin{bmatrix} .7 & .3 \\ 2 & 0 \end{bmatrix}$$

An application associated with this matrix is a simple model of a growing bird population. Let

$$x_n = \text{ no. of adults in year } n,$$

$$y_n = \text{ no. of juveniles in year } n.$$

We have a matrix equation to represent changes from year to year. We have 30% of the juveniles survive to become adults, 70% of the adults survive a year, and each adult has 2 offspring (juveniles). We have this information summarized in a matrix equation:

$$\begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix} = \begin{bmatrix} .7 & .3 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} x_n \\ y_n \end{bmatrix}.$$

We deduce, by induction, that

$$\begin{bmatrix} x_n \\ y_n \end{bmatrix} = A^n \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}.$$

This is a sample of many applications where we wish to know what happens to A^n as $n \longrightarrow \infty$.

Recall our computation of eigenvalues/eigenvectors for this matrix:

First we define an eigenvector $\mathbf{x}$ of eigenvalue λ to be satisfy $A\mathbf{x} = \lambda\mathbf{x}$ and $\mathbf{x} \neq \mathbf{0}$. This is equivalent to $(A - \lambda I)\mathbf{x} = \mathbf{0}$ and $\mathbf{x} \neq \mathbf{0}$. This can only occur by our previous observations when $\det(A - \lambda I) = 0$ and moreover when $\det(A - \lambda I) = 0$ we can find an $\mathbf{x} \neq \mathbf{0}$ with $A\mathbf{x} = \lambda\mathbf{x}$.

$$\begin{aligned} \det(A - \lambda I) &= \det(\begin{bmatrix} .7-\lambda & .3 \\ 2 & -\lambda \end{bmatrix}) \\ &= (.7 - \lambda)(-\lambda) - .3 \times 2 \\ &= \frac{1}{10}(10\lambda^2 - 7\lambda - 6) \\ &= \frac{1}{10}(5\lambda - 6)(2\lambda + 1) \end{aligned}$$

Thus we have two eigenvalues $\lambda = \frac{6}{5}, \frac{-1}{2}$.

For $\lambda = \frac{6}{5}$, we solve $(A - \frac{6}{5}I)\mathbf{v} = \mathbf{0}$ for $\mathbf{v} \neq \mathbf{0}$:

$$(A - \frac{6}{5}I)\mathbf{v} = \begin{bmatrix} -.5 & .3 \\ 2 & -1.2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The vector $\mathbf{v} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$ works as an eigenvalue of A of eigenvalue $\frac{6}{5}$. We check

$$\begin{bmatrix} .7 & .3 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 3 \\ 5 \end{bmatrix} = \begin{bmatrix} 3.6 \\ 6 \end{bmatrix} = \frac{6}{5} \begin{bmatrix} 3 \\ 5 \end{bmatrix}.$$

For $\lambda = \frac{-1}{2}$, we solve $(A - \frac{-1}{2}I)\mathbf{v} = \mathbf{0}$ for $\mathbf{v} \neq \mathbf{0}$:

$$(A - \frac{-1}{2}I)\mathbf{v} = \begin{bmatrix} 1.2 & .3 \\ 2 & .5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The vector $\mathbf{v} = \begin{bmatrix} 1 \\ -4 \end{bmatrix}$ works as an eigenvalue of A of eigenvalue $\frac{-1}{2}$. We check

$$\begin{bmatrix} .7 & .3 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ -4 \end{bmatrix} = \begin{bmatrix} -.5 \\ 2 \end{bmatrix} = \frac{-1}{2} \begin{bmatrix} 1 \\ -4 \end{bmatrix}.$$

Note that we will always succeed in finding an eigenvector (a non zero vector) assuming our eigenvalue λ has $\det(A - \lambda I) = 0$. Thue if you are doing such a computation and find that you are unable to find a non zero vector $\mathbf{x}$ with $(A - \lambda I)\mathbf{x} = \mathbf{0}$, then either you made an error determining the eigenvalues or you made an error solving for the non zero vector $\mathbf{x}$ from the equations $(A - \lambda I)\mathbf{x} = \mathbf{0}$. I have seen both errors from students.

There is one example which needs to be given, namely the case of a 0 eigenvalue.

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$$

We compute $\det(A - \lambda I) = \det(\begin{bmatrix} 1 - \lambda & 2 \\ 2 & 4 - \lambda \end{bmatrix}) = \lambda^2 - 5\lambda$. One of the roots is $\lambda = 0$.

For $\lambda = 0$, we solve for $(A - 0I)\mathbf{v} = \mathbf{0}$ for some $\mathbf{v} \neq \mathbf{0}$. In this case we find $\mathbf{v} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$ works with $A\mathbf{v} = 0 \cdot \mathbf{v} = \mathbf{0}$. This is a bit confusing since $\mathbf{v}$ also shows that A is not invertible but don't lose sight of the fact that $\mathbf{v}$ is a happy eigenvector of eigenvalue 0. The given matrix also has eigenvalue 5 with eigenvector $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$. Check this.