Notes on a Basic Business Problem
MATH 104 and MATH 184
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This simple problem will introduce you to the basic ideas of revenue, cost, profit, and demand. These are not so hard, but you need to build a vocabulary so that you can use these terms properly without second-guessing yourself.

Demand can be a difficult concept when you first meet it. Demand is the relationship between the price of an item and the number of units that will sell at that price. It is a sociological relationship in that it is rooted in the behaviour of consumers. It is a basic principal of economics that the demand relationship has the characteristic that an increase in price will lead to a decrease in demand. The simplest such relationship is a linear one, and in the first problem we present, we will use a linear demand relationship. In general, demand relationships are non-linear. In practice, determining the "real" demand relationship for a product is non-trivial, but in this course you will always be given this relationship, or enough information to determine it. (You may wish to consider how realistic some of the demand relationships that we give you are.)

It is usual to use the variables p for the price of a unit, and q for the quantity demanded.

In the problem below, we will plot the demand relationship on the (q, p)-plane and treat p as a function of q. However, it is important to note that q is not really an *independent* variable in these problems (except mathematically). Moreover, although the producer has the ability to set the price p, the demand relationship is NOT in control of the producer, so setting p determines how many items q will be sold. (In the strictest sense, we are in the situation of a *monopoly producer*, but there is not need to dwell on this in this course.)

Many (but not all!) of you are taking a first year economics course, so you will be learning these terms in that course as well. You will discover that there will be differences over the term between the presentation in this course and that in economics; mostly this arises because first year economics courses do not have a calculus pre-requisite. Usually these differences come from either conventions or from choices about which quantities are studied in problems. These can make for some interesting points of discussion in class.

We will introduce a basic business problem as an example. Note that some brief notes on the business concepts in this problem appear after it in these notes. I'd like to thank my summer student Patrick Chan, a senior BCOM student, for his assistance in preparing these later notes.

A Basic Business Problem

Opple Inc. is the only manufacturer of the popular oPad. Opple estimates that when the price of the oPad is $200, then the weekly demand for it is 5000 units. For every $1 increase in the price, the weekly demand decreases by 50 units. Assume that the fixed costs of production on a weekly basis are $100 000, and the variable costs of production are $75 per unit.

(a) Find the linear demand equation for the oPad. Use the notation p for the unit price and q for the weekly demand.

ANS : $p = -\dfrac{1}{50}q + 300$

(b) Find the weekly cost function, $C = C(q)$, for producing q oPads per week. Note that $C(q)$ is a linear function.

ANS : $C(q) = 100000 + 75q$

(c) Find the weekly revenue function, $R = R(q)$. Note that $R(q)$ is a quadratic function.

ANS: $R(q) = p \cdot q = q\left(300 - \dfrac{1}{50}q\right).$

Note that this revenue relationship "Revenue = price per unit times quantity demanded" is central in your studies in business. While it looks simple, we will run into situations throughout the term where this relationship will produce counterintuitive results for you because p and q are related by a demand relationship. For example, sometimes increasing the price will cause a decrease in revenue.

(d) The *break-even* points are where Cost equals Revenue; that is, where $C(q) = R(q)$. Find the break-even points for the oPad.

ANS: Solve the quadratic you get by setting $R = C$.

(e) On the same set of axes, sketch graphs of $C = C(q)$ and $R = R(q)$ and use these graphs to help you explain why there are two break-even points.

ANS: Make a reasonable sketch, with appropriate points labelled.

(f) *Profit* is defined as Revenue minus Cost: $P(q) = R(q) - C(q)$. Find the profit function $P(q)$. Note that it is a quadratic function.

ANS: This is a simple calculation.

(g) Graph $P = P(q)$ on the same axes as you sketched the graphs of $C(q)$ and $R(q)$. On this graph, indicate the regions of profit $(P(q) > 0)$ and loss $(P(q) < 0)$.

ANS: The easiest way to indicate it is to point out the interval on the q-axis where $P > 0$.

(h) How should Opple Inc. operate in order to maximize the weekly profit $P = P(q)$? Use mathematics in your explanation.

ANS: Find the vertex of the Profit quadratic. One might also use calculus to solve this problem (which is a bit of overkill for this quadratic function). Those of you in Math 104 will know this procedure, at least at this elementary level. in Math 184, you will learn about this maximization procedure in more detail later in the course, so in the meantime, try to understand it graphically. When we look at more complicated demand relationships, we will need a procedure that will work on other functions in addition to quadratics. Fortunately, calculus will provide us with such a procedure.

Some Notes on Revenue, Cost, and Profit
compiled by Patrick Chan
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Marginal Cost:
Common Economics Definition: The additional cost of producing 1 extra unit.

Formal Definition: The derivative of cost with respect to quantity $(\frac{dC}{dq})$.

EXAMPLE $C(q) = 2000 + 10q^2$

$$\text{Marginal Cost} = \frac{dC}{dq} = 20q$$

Fixed Costs:
Definition: Costs of production that do not change with production (Quantity).

EXAMPLE $C(q) = 2000 + 10q^2$

$$\text{Fixed Costs} = 2000$$

Variable Costs: (Note: **Not the Same as MARGINAL COSTS!**)

Definition: Costs that vary with quantity (anything with the term q in the cost function).

EXAMPLE $C(q) = 2000 + 10q^2$

$$\text{Variable Cost} = 10q^2$$

Marginal Revenue

Common Economics Definition: Additional Revenue for 1 extra unit of production.

Formal Definition: Marginal Revenue $= \dfrac{dR}{dq}$.

We know that revenue (R) is computed as Price x Quantity $(p * q)$:

$$\boxed{R = pq.}$$

EXAMPLE Let $p = 30 - 5q$ be the demand equation.
Since we know that $R = pq$

$$R = 30q - 5q^2. \tag{1}$$

The derivative of the revenue function with respect to quantity will be

$$\frac{dR}{dq} = 30 - 10q. \tag{2}$$

Profit
Quite simply, Profit = Revenue - Cost.
We will use P to represent Profit so be careful not confuse it with p, which is Price. This gives us the basic equation

$$P = R - C. \tag{3}$$

Profit is maximized at the level of output where marginal profit is zero. In economics, this is usually taken to mean that an additional unit of output q will not change the profit. In this course, we will use the derivative definition of marginal and so profit is maximized when

$$\frac{dP}{dq} = 0. \tag{4}$$

Taking the derivative on both sides, we obtain

$$\frac{dP}{dq} = \frac{dR}{dq} - \frac{dC}{dq} = 0, \tag{5}$$

which means

$$MR - MC = 0. \tag{6}$$

That is, we will examine output values q at which marginal revenue equals marginal cost:

$$\boxed{MR = MC.} \tag{7}$$

This means we will solve for q after setting $MR = MC$

EXAMPLE
Find the optimal output in order to maximize profit given:

$$\frac{dC}{dq} = 8q, \tag{8}$$

$$\frac{dR}{dq} = 900 - 10q. \tag{9}$$

SOLUTION

Set $\dfrac{dC}{dq} = \dfrac{dR}{dq}$:

$$8q = 900 - 10q. \tag{10}$$

Now solve for q.

$$18q = 900, \tag{11}$$

so

$$\boxed{q = 50.} \tag{12}$$

How do you know this is a *maximum*? (The answer lies either in some high school algebra r in an argument using calculus.)

EXAMPLE

Find the optimal output in order to maximize profit given that revenue function is $R(q) = 9000Q - 207q^2$ and the cost function is $C(q) = 18q^2$.

SOLUTION

We need to find MR and MC first.

$$\frac{dC}{dq} = 36q \tag{13}$$

and

$$\frac{dR}{dq} = 9000 - 414q. \tag{14}$$

Equating (13) and (14), we get

$$36q = 9000 - 4141, \tag{15}$$

which gives

$$\boxed{q = 20.} \tag{16}$$

Demand Curve

Definition: The *demand curve* is a curve or schedule (table of values) showing the total quantity of a good of uniform quality that buyers want to buy at each price during a particular period of time provided that all other things are held constant. Mathematically, this means it is a relation between the price p and the quantity demanded q.

Linear demand means this relation is a straight line; *viz.*

$$q = a - bp, \tag{17}$$

where q is the quantity demanded, p is the demand price, and a and b are given parameters (constants).

The so-called Law of Demand says that any real demand curve is always downwards sloping when the vertical axis is price p and the horizontal axis is demand quantity q. This means that if price is increased, demand will fall. Note that mathematically, this labelling of the axes is arbitrary (though seems to be a matter for strong opinions amongst some!).

EXAMPLE
Suppose the demand curve for a product produced by a firm is given by $q = 1350 - 5p$ and the cost function is $C(q) = 60q + 4q^2$. Find the profit maximizing output for the firm.

SOLUTION
We will need to find the marginal revenue MR using the given demand function.

Recall that $R = pq$. If we isolate for p in the demand equation, we can multiply everything by q to find the revenue function:

$$5p = 1350 - q, \tag{18}$$

which yields

$$p = 270 - \frac{q}{5}. \tag{19}$$

Now,

$$R = pq \tag{20}$$

and so, as a function of q,

$$R(q) = 270q - \frac{q^2}{5}. \tag{21}$$

This means that marginal revenue is

$$\frac{dR}{dq} = 270 - \frac{2q}{5}. \tag{22}$$

The marginal cost $\frac{dC}{dq}$ is

$$\frac{dC}{dq} = 60 + 8q. \tag{23}$$

Setting $\frac{dC}{dq} = \frac{dR}{dq}$, we get

$$270 - 0.4q = 60 + 8q, \tag{24}$$

which gives

$$q = 25. \tag{25}$$

How do we know this value of q gives us a *maximum* profit?

We are interested often in the *break-even* points. That is, these are the points at which the revenues balance the costs and we make no profit. We want to understand these points since they help us decide our production behaviour since varying production when we are close to a break-even point can mean the difference between profitability and generating a loss.

Mathematically, to find the break even points, we just set $R(q) = C(q)$ and solve for q. In this example, this gives

$$270q - \frac{q^2}{5} = 60q + 4q^2. \tag{26}$$

Solving gives us

$$4.2q^2 = 210q, \tag{27}$$

or

$$\boxed{q = 0 \text{ or } 50.} \tag{28}$$

Note that $q = 0$ would mean the firm produces nothing, and therefore has no revenue and no cost (note that our cost function does not include fixed costs, which is a tad unrealistic). Hence, the more interesting break-even point is to produce $q = 50$. What happens if we increase q from this value? If we decrease it?