MATH 184 Exponential and Logarithm Rules

Many students who have trouble with algebra of polynomials have even more trouble with the special case of exponentials and logarithms. We are typically introduced to the function 2^x or 10^x and then the logarithms are introduced as the compositional inverses. Thus we have

$$f(x) = 2^x \quad , \quad g(x) = \log_2(x) \text{ with } g(f(x)) = x \text{ and } f(g(x)) = x \text{ for } x > 0$$

We note that the domain of $f(x)$ is all $\mathbf{R}$ and the range (or image) of f is $\{x \in \mathbf{R} : x > 0\}$. thus the domain of $g(x) = \log_2(x)$ is $\{x \in \mathbf{R} : x > 0\}$ and the range is $\mathbf{R}$.

At this stage without an explicit definition of e, we can just think of it as a number (about 2.7) and we denote $\log_e(x)$ as $\ln(x)$. For the rules below we are using an arbitrary base a although the bulk of the exponential and logarithm functions you will see have $a = e$.

Our properties/rules come in pairs. Assume $a > 0$.

$$a^0 = 1 \qquad \text{yields} \qquad \log_a(1) = 0$$

We have for $b, c > 0$

$$a^{y+z} = a^y a^z \qquad \text{yields} \qquad \log_a(bc) = \log_a(b) + \log_a(c).$$

We obtain this by taking $\log_a$ of both sides of $a^{y+z} = a^y a^z$ to obtain $\log_a(a^{y+z}) = \log_a(a^y a^z)$. Of course $\log_a(a^{y+z}) = y + z$. Now if we let $b = a^y$ and $c = a^z$, then $bc = a^{y+z}$ while $y = \log_a(b)$ and $z = \log_a(c)$. Thus we have $\log_a(bc) = y + z = \log_a(b) + \log_a(c)$. This idea of this variable substitution will be a little hard for you. In particular for any choice $b > 0$, there will be a y so that $a^y = b$, and in fact $y = \log_a(b)$. Please note that I am not asking you to reproduce these arguments but instead showing you where the rules come from.

We have for $b > 0$

$$a^{-y} = \frac{1}{a^y} \qquad \text{yields} \qquad \log_a(\frac{1}{b}) = -\log_a(b)$$

We obtain this by taking $\log_a$ of both sides of $a^{-y} = \frac{1}{a^y}$ to obtain $\log_a(a^{-y}) = \log_a(\frac{1}{a^y})$ and so $-y = \log_a(\frac{1}{a^y})$. Substituting $b = a^y$ so that $\log_a(b) = y$ we have $\log_a(\frac{1}{b}) = -y = -\log_a(b)$.

We have

$$(a^y)^z = a^{yz} \qquad \text{yields} \qquad \log_a(c) = \log_a(b)\log_b(c).$$

We obtain this by taking $\log_a$ of both sides of $(a^y)^z = a^{yz}$ to obtain $\log_a((a^y)^z) = xy$. Now $x = \log_a(a^x)$ and $y = \log_{a^x}((a^x)^y)$ (note that $a^x > 0$ so that $\log_{a^x}$ is defined). If we let $b = a^x$ and $c = (a^x)^y$ we have $\log_a(c) = \log_a((a^x)^y) = xy = \log_a(b)\log_b(c)$. The change of logarithm base is always a little hard to remember!

Of course we could derive the exponent laws from the logarithm laws.