The Inverse Trigonometric Functions

These notes amplify on the book's treatment of inverse trigonometric functions and supply some needed practice problems. Please see pages 543–544 for the graphs of $\sin^{-1} x$, $\cos^{-1} x$, and $\tan^{-1} x$.

1 The Arcsine Function

My sine is x; who am I? If x is any real number such that $|x| \leq 1$, there are infinitely many possible answers. For example, let $x = 1/2$. Then $\sin y = x$ when $y = \pi/6$, $5\pi/6$, $-7\pi/6$, $-11\pi/6$, $13\pi/6$, $17\pi/6$, $-19\pi/6$, and so on.

Note however that as y travels from $-\pi/2$ to $\pi/2$, $\sin y$ travels from -1 to 1. Since $\sin y$ is continuous and increasing in the interval $-\pi/2 \leq y \leq \pi/2$, it follows that for any x between -1 and 1 there is exactly one y between $-\pi/2$ and $\pi/2$ such that $\sin y = x$. We can therefore define a new function $\sin^{-1}$ as follows:

Definition 1. If $-1 \leq x \leq 1$, then $\sin^{-1} x$ (also known as $\arcsin x$) is the number between $-\pi/2$ and $\pi/2$ whose sine is equal to x.

Comment. If the function f has an inverse, that inverse is generally denoted by f^{-1}. The notation $\sin^{-1} x$ that has just been introduced is (more or less) in accord with this general convention. Not quite! The sine function *does not* have an inverse, since given $\sin t$ it is not possible to recover t uniquely.

Maybe we are being too fussy. But the notation can also be a source of confusion. For note that $\sin^2 x$ is a standard abbreviation for $(\sin x)^2$, and $\sin^3 x$ is a standard abbreviation for $(\sin x)^3$. So should $\sin^{-1} x$ mean $(\sin x)^{-1}$? Maybe it should. *But it doesn't!*

The notation $\arcsin x$ is preferred by many mathematicians. Unfortunately, $\sin^{-1}$ seems to be gaining ground over arcsin, maybe because it fits the cramped space on calculator keyboards better. There are several other notations, including "Arc sin," and "asin."

Pronunciations vary: $\sin^{-1} x$ can be pronounced "sine inverse (of) x," "inverse sine (of) x," or even "arc sine x."

Comment. You don't have to know a lot about geography to know the capital of the country whose capital is Amman. And you don't have to know much about trigonometry to find $\sin(\sin^{-1}(0.123))$. Indeed it is clear that

$$\sin(\sin^{-1} x) = x \quad \text{for all } x \text{ between } -1 \text{ and } 1.$$

The behaviour of $\sin^{-1}(\sin x)$ is more complicated. Let x be any number in the interval $[-\pi/2, \pi/2]$. Then the number between $-\pi/2$ and $\pi/2$ whose sine is $\sin x$ is clearly x.

But suppose for example that $\pi/2 \le x \le 3\pi/2$. Since the sine function is symmetrical about the line $x = \pi/2$, we have $\sin x = \sin(\pi - x)$. And since $\pi - x$ lies between $-\pi/2$ and $\pi/2$,

$$\sin^{-1}(\sin x) = \sin^{-1}(\sin(\pi - x)) = \pi - x.$$

2 Differentiating the Arcsine Function

Let $y = \sin^{-1} x$. Then for all x in the interval $[-1, 1]$

$$\sin y = x.$$

Assume that y is differentiable at all x in $(-1, 1)$—it really is, but we omit the proof. If we differentiate both sides of the equation above with respect to x, then the Chain Rule gives

$$(\cos y)\frac{dy}{dx} = 1.$$

Thus if $\cos y \neq 0$ then

$$\frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\cos(\sin^{-1} x)}.$$

The above formula is correct but unattractive. To improve it, recall the familiar identity

$$\cos^2 y + \sin^2 y = 1.$$

Since $y = \sin^{-1} x$, we have

$$\cos^2 y = 1 - (\sin y)^2 = 1 - (\sin(\sin^{-1} x))^2 = 1 - x^2.$$

But y lies in the interval $(-\pi/2, \pi/2)$, and therefore $\cos y$ is positive. It follows that $\cos y = \sqrt{1-x^2}$ and therefore

$$\boxed{\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}} \tag{1}$$

3 The Arctangent Function

The tangent function is continuous and increasing on the interval $(-\pi/2, \pi/2)$. Moreover, as t approaches $-\pi/2$ from the right, $\tan t$ becomes arbitrarily large negative, while as t approaches $\pi/2$ from the left, $\tan t$ becomes arbitrarily large positive. It follows that as t ranges over $(-\pi/2, \pi/2)$, $\tan t$ takes on every real value exactly once.

Thus the tangent function, restricted to the interval $(-\pi/2, \pi/2)$, has an inverse. We can therefore define a new function $\tan^{-1}$ as follows:

Definition 2. For any real number x, $\tan^{-1} x$ (also known as $\arctan x$) is the number between $-\pi/2$ and $\pi/2$ whose tangent is equal to x.

4 Differentiating the Arctangent Function

Let $y = \tan^{-1} x$. Then for all x

$$\tan y = x.$$

Assume that y is differentiable for all x—it really is, but we omit the proof. If we differentiate both sides of the above equation with respect to x, we obtain

$$(\sec^2 y)\frac{dy}{dx} = 1.$$

It follows that

$$\frac{dy}{dx} = \frac{1}{\sec^2 y} = \frac{1}{\sec^2(\tan^{-1} x)}.$$

To make the above formula more attractive, recall the identity $1 + \tan^2 y = \sec^2 y$. If you do not recall it, you can quickly derive it by dividing both sides of the identity $\cos^2 y + \sin^2 y = 1$ by $\cos^2 y$.

Since $y = \tan^{-1} x$, we have

$$\sec^2 y = 1 + (\tan y)^2 = 1 + (\tan(\tan^{-1} x))^2 = 1 + x^2,$$

and therefore we can rewrite the formula for the derivative of $\tan^{-1} x$ as

$$\boxed{\frac{d}{dx}(\tan^{-1} x) = \frac{dy}{dx} = \frac{1}{1+x^2}} \tag{2}$$

5 The Arccosine Function

The cosine function decreases from 1 to -1 over the interval $[0, \pi]$. It is therefore reasonable to define $\cos^{-1} x$ to be the number between 0 and π whose cosine is x. By an argument almost identical to the argument in Section 2, we can show that

$$\frac{d}{dx}(\cos^{-1} x) = \frac{dy}{dx} = \frac{-1}{\sqrt{1-x^2}}. \tag{3}$$

There is an easier way to derive Equation 3. Note that $\cos y = \sin(\pi/2 - y)$ for all y, and that as y travels from 0 to π, $\pi/2 - y$ travels from $\pi/2$ down to $-\pi/2$. It follows that $\cos^{-1} x = \pi/2 - \sin^{-1} x$. Differentiate both sides with respect to x: we get Equation 3. Since $\cos^{-1} x$ is such a close relative of $\sin^{-1} x$, the function $\cos^{-1} x$ occurs explicitly in very few formulas.

6 Other Inverse Trigonometric Functions

We could also define the inverse trigonometric functions $\sec^{-1} x$, $\csc^{-1} x$, and $\cot^{-1} x$. We differentiate $\sec^{-1} x$, partly because it is the only one of the three that gets seriously used, but mainly as an exercise in algebra.

Define $\sec^{-1} x$ as the number between 0 and π whose secant is x. We could differentiate $\sec^{-1} x$ by using an approach along the lines of Section 2, but there is a much easier way.

If an angle has secant equal to x, then it has cosine equal to $1/x$. Thus $\sec^{-1} x = \cos^{-1}(1/x)$. Differentiate both sides with respect to x, using the Chain Rule and Equation 3. We obtain

$$\frac{d}{dx}(\sec^{-1} x) = \frac{1}{x^2\sqrt{1-\frac{1}{x^2}}}. \tag{4}$$

We can simplify Equation 4, but it has to be done carefully, since the square root of x^2 is *not* equal to x when x is negative. We have

$$x^2\sqrt{1-\frac{1}{x^2}} = x^2\sqrt{\frac{x^2-1}{x^2}} = x^2\frac{\sqrt{x^2-1}}{\sqrt{x^2}} = x^2\frac{\sqrt{x^2-1}}{|x|} = |x|\sqrt{x^2-1}$$

and therefore

$$\frac{d}{dx}(\sec^{-1} x) = \frac{1}{|x|\sqrt{x^2-1}}. \tag{5}$$

Note. There are other definitions of $\sec^{-1} x$: everyone agrees that when x is positive then $\sec^{-1} x$ should lie between 0 and $\pi/2$. But for negative x, some people define $\sec^{-1} x$ to be the number between π and $3\pi/2$ whose secant is x. Under this definition, the derivative of $\sec^{-1} x$ turns out to be $1/(x\sqrt{x^2-1})$.

7 Integrating the Inverse Trigonometric Functions

The differentiation formulas 1 and 2 can be rewritten as integration formulas:

$$\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x + C$$

and

$$\int \frac{dx}{1+x^2} = \tan^{-1} x + C.$$

These integration formulas explain why the calculus 'needs' the inverse trigonometric functions. The functions $1/\sqrt{1-x^2}$, $1/(1+x^2)$, and their close relatives come up naturally in many applications. The inverse trigonometric functions supply names for the antiderivatives of these important functions.

8 Problems

1. (a) Graph the curve $y = \sin^{-1}(\sin x)$ for $-4\pi \le x \le 4\pi$.

   (b) Graph the curve $y = \arccos(\cos x)$ for $-4\pi \le x \le 4\pi$.

2. Graph the curve $y = \tan^{-1}(\tan x) - \tan(\tan^{-1} x)$ for $-4\pi \le x \le 4\pi$.

3. Let $f(x) = \sin(\tan^{-1} x)$. Find a formula for $f(x)$ that does not involve trigonometric functions.

4. Let $y = \frac{x}{2}\arcsin x + \frac{x}{2}\sqrt{1-x^2}$. Find $\frac{dy}{dx}$ and simplify as much as possible.

5. Let $y = 2x\tan^{-1} x - \ln(1+x^2)$. Find $\frac{dy}{dx}$.

6. Let $f(t) = \sin^{-1}(\sqrt{1-t})$. Find $f'(t)$ and simplify.

7. Let $f(u) = (\arcsin u)^{-1}$. Find $f'(1/\sqrt{2})$.

8. Let $y = \left(\frac{1}{3}\right)\tan^{-1}\left(\frac{x}{3}\right)$. Find $\frac{dy}{dx}$ and simplify.

9. Find the equation of the tangent line to $y = \arctan(x^2)$ at the point on the curve which has x-coordinate equal to 1.

10. Find the equations of the two lines that are tangent to the curve $y = \sin^{-1} x$ and have slope equal to $\sqrt{2}$.

11. Show that the curve with equation $\tan^{-1} x + \tan^{-1} y = \pi/2$ passes through the point $(1/\sqrt{3}, \sqrt{3})$, and find the equation of the tangent line to the curve at this point.

12. Show that if $\ln(x^2 + y^2) + 2\tan^{-1}\frac{x}{y} = 0$ and $x \neq y$ then $\frac{dy}{dx} = \frac{x+y}{x-y}$.

13. Let $y = 2^{\tan^{-1} x}$. Find $\frac{dy}{dx}$.

14. Let $f(t) = \arctan(\sqrt{t^2 - 1})$. Find $f'(t)$.

15. Let $f(x) = \tan^{-1}\left(\frac{x+1}{x-1}\right) + \tan^{-1} x$. Find $f'(x)$ and simplify.

16. Let $\cot^{-1} x$ be the number in the interval $[0, \pi]$ whose cotangent is x. Find the derivative of $\cot^{-1} x$ with respect to x.

17. Calculate the following integrals:

    (a) $\displaystyle\int \frac{dx}{1+4x^2}$; (b) $\displaystyle\int \frac{dx}{\sqrt{4-x^2}}$; (c) $\displaystyle\int \frac{x\,dx}{4+9x^4}$.

18. Calculate the following integrals:

    (a) $\displaystyle\int_{-1/2}^{1/2} \arcsin x\,dx$; (b) $\displaystyle\int_{1}^{\sqrt{3}} \arctan x\,dx$; (c) $\displaystyle\int_{0}^{\infty} \arctan(3x+1)\,dx$.

19. Calculate the following integrals

    (a) $\displaystyle\int x \arctan x\,dx$; (b) $\displaystyle\int x \sin^{-1} x\,dx$; (c) $\displaystyle\int \ln(1+x^2)\,dx$.

20. Find, exactly, the area of the region that lies below $y = \frac{1}{1+4x^2}$ but above $y = \frac{1}{4+x^2}$.