# Forbidden Configurations: Boundary Cases

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Consider the following family of subsets of $\{1,2,3,4\}$:

$$\mathcal{A} = \{\emptyset, \{1,2,4\}, \{1,4\}, \{1,2\}, \{1,2,3\}, \{1,3\}\}$$

The incidence matrix $A$ of the family $\mathcal{A}$ of subsets of $\{1,2,3,4\}$ is:

$$A = \begin{bmatrix} 0 & 1 & \boxed{1} & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 \end{bmatrix}$$

**Definition** We say that a matrix $A$ is *simple* if it is a (0,1)-matrix with no repeated columns.

**Definition** We define $\|A\|$ to be the number of columns in $A$.

$$\|A\| = 6 = |\mathcal{A}|$$

**Definition** Given a matrix $F$, we say that $A$ has $F$ as a *configuration* if there is a submatrix of $A$ which is a row and column permutation of $F$.

$$F = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix} \quad \in \quad A = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & \boxed{1 \quad 0 \quad 1} & & & 1 & \boxed{0} \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & \boxed{1 \quad 1 \quad 0} & & & 0 & \boxed{0} \end{bmatrix}$$

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$$F = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix} \quad \in \quad A = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & \boxed{1 \;\; 0 \;\; 1} & & 1 & \boxed{0} \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & \boxed{1 \;\; 1 \;\; 0} & & 0 & \boxed{0} \end{bmatrix}$$

We consider the property of forbidding a configuration $F$ in $A$.

**Definition** Let

*forb*$(m, F)$= $\max\{\|A\| : A \ m\text{-rowed simple, no configuration } F\}$

# A Product Construction

The building blocks of our product constructions are $I$, $I^c$ and $T$, e.g:

$$I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad I_4^c = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}, \quad T_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

# The Conjecture

We conjecture that our product constructions with the three building blocks $\{I, I^c, T\}$ determine the asymptotically best constructions.

**Definition** Given two matrices $A, B$, we define the product $A \times B$ as the matrix whose columns are obtained by placing a column of $A$ on top of a column of $B$ in all possible ways. (A, Griggs, Sali 97)

$$\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} \times \begin{bmatrix}1&1&1\\0&1&1\\0&0&1\end{bmatrix} = \left[\begin{array}{ccc|ccc|ccc}1&1&1&0&0&0&0&0&0\\0&0&0&1&1&1&0&0&0\\0&0&0&0&0&0&1&1&1\\\hline 1&1&1&1&1&1&1&1&1\\0&1&1&0&1&1&0&1&1\\0&0&1&0&0&1&0&0&1\end{array}\right]$$

Given $p$ simple matrices $A_1, A_2, \ldots, A_p$, each of size $m/p \times m/p$, the $p$-fold product $A_1 \times A_2 \times \cdots \times A_p$ is a simple matrix of size $m \times (m/p)^p$ i.e. $\Theta(m^p)$ columns.

# Examples

$$[01] \times [01] = K_2 = \left[ \begin{array}{cccc} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{array} \right]$$

$I_{m/2} \times I_{m/2}$ is vertex-edge incidence matrix of $K_{m/2,m/2}$

# The Conjecture

We conjecture that our product constructions with the three building blocks $\{I, I^c, T\}$ determine the asymptotically best constructions.

**Definition** Let $F$ be given. Let $x(F)$ denote the largest $p$ such that there is a $p$-fold product which does not contain $F$ as a configuration where the $p$-fold product is $A_1 \times A_2 \times \cdots \times A_p$ where each $A_i \in \{I_{m/p}, I^c_{m/p}, T_{m/p}\}$.

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The conjecture has been verified for $k \times \ell$ $F$ where $k = 2$ (A, Griggs, Sali 97) and $k = 3$ (A, Sali 05) and $\ell = 2$ (A, Keevash 06) and for $k$-rowed $F$ with bounds $\Theta(m^{k-1})$ or $\Theta(m^k)$ (A, Fleming 10) plus other cases.

# Boundary Cases

**Definition** Let $F$ be a $k$-rowed configuration and let $\alpha$ be a $k$-rowed column vector. Define $[F|\alpha]$ to be the concatenation of $F$ and $\alpha$.

**Definition** Let $F$ be a $k$-rowed configuration. We say that $F$ is a boundary case if for every $k$-rowed column $\alpha$ which is either

not present in $F$

or

present just once in $F$,

then $forb(m,[F|\alpha])$ is $\Omega(m \cdot forb(m,F))$.

# Example of a Boundary Case

$$F = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix}, \quad forb(m,F) = 2m$$

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Construction: $I^c_{m/2} \times I^c_{m/2}$ avoiding $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

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$$[F|\alpha] = \left[\begin{array}{cc|ccc} 0 & 0 & 0 & 1 & \color{red}{1} \\ 1 & 0 & 1 & 0 & \color{red}{1} \\ 0 & 1 & 1 & 1 & \color{red}{0} \end{array}\right], \quad forb(m, [F|\alpha]) \text{ is } \Omega(m^2)$$

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# Example of a Boundary Case

$$F = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix}, \quad forb(m, F) = 2m$$

Thus $F$ is a boundary case.

Using a result of A and Fleming 10, there are three simple column-maximal 4-rowed $F$ for which $forb(m, F)$ is quadratic. Here is one example:

$$F_8 = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{bmatrix}$$

How can we repeat columns in $F_8$ and still have a quadratic bound? We note that repeating either the column of sum 1 or the column of sum 3 will result in a cubic lower bound. Thus we only consider taking multiple copies of the columns of sum 2.

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How can we repeat columns in $F_8$ and still have a quadratic bound? We note that repeating either the column of sum 1 or the column of sum 3 will result in a cubic lower bound. Thus we only consider taking multiple copies of the columns of sum 2. For a fixed $t$, let

$$F_8(t) = \left[ \begin{matrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{matrix} \; t \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \\ 0 & 0 \end{bmatrix} \right]$$

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**Theorem** (A, Raggi, Sali 09) *Let $t$ be given. Then forb$(m, F_8(t))$ is $\Theta(m^2)$. Moreover $F_8(t)$ is a boundary case, namely for any column $\alpha$ not already present $t$ times in $F_8(t)$, then forb$(m, [F_8(t)|\alpha])$ is $\Omega(m^3)$.*

The proof of the upper bound is currently a rather complicated induction with some directed graph arguments.

# $5 \times 6$ Simple Configuration with Quadratic bound

The Conjecture predicts nine 5-rowed simple matrices $F$ to be boundary cases, namely *forb*$(m, F)$ is predicted to be $\Theta(m^2)$ and for any column $\alpha$ we have *forb*$(m, [F|\alpha])$ being $\Omega(m^3)$. We have handled the following case.

$$F_7 = \begin{bmatrix} 1 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

**Theorem** (A, Raggi, Sali) *forb*$(m, F_7)$ *is* $\Theta(m^2)$. *Moreover* $F_7$ *is a boundary case, namely for any column* $\alpha$, *then forb*$(m, [F_7|\alpha])$ *is* $\Omega(m^3)$.

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$$F_7 = \begin{bmatrix} 1 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

**Theorem** (A, Raggi, Sali) *$forb(m, F_7)$ is $\Theta(m^2)$. Moreover $F_7$ is a boundary case, namely for any column $\alpha$, then $forb(m, [F_7|\alpha])$ is $\Omega(m^3)$.*

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# All 6-rowed Configurations with Quadratic Bounds

$$G_{6\times 3} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

**Theorem** (A,Raggi,Sali) *forb*$(m, G_{6\times 3})$ *is* $\Theta(m^2)$. *Moreover* $G_{6\times 3}$ *is a boundary case, namely for any column* $\alpha$, *then* *forb*$(m, [G_{6\times 3}|\alpha])$ *is* $\Omega(m^3)$. *In fact if* $F$ *is not a configuration in* $G_{6\times 3}$, *then* *forb*$(m, F)$ *is* $\Omega(m^3)$.

**Proof:** We use induction and the bound for $F_7$.

**Theorem** (Balogh and Bollabás 05) *Given* $k$*, there exists a constant* $c_k$ *so that* $forb(m, \{I_k, I_k^c, T_k\}) = c_k$.

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**Theorem** (A. and Meehan 11) *Let* $p, k$ *be given with* $p \geq 3k$*.*
*Let* $F = [\mathbf{0}_k | I_k] \times [\mathbf{0}_k | T_k] \times [I_k^c | \mathbf{1}_k] \times K_{p-3k}$*.*
*Then* $forb(m, F)$ *is* $\Theta(m^{p-k})$*.*

**Theorem** (Balogh and Bollabás 05) *Given $k$, there exists a constant $c_k$ so that forb$(m, \{I_k, I_k^c, T_k\}) = c_k$.*

**Theorem** (A. and Meehan 11) *Let $p, k$ be given with $p \geq 3k$. Let $F = [\mathbf{0}_k|I_k] \times [\mathbf{0}_k|T_k] \times [I_k^c|\mathbf{1}_k] \times K_{p-3k}$. Then forb$(m, F)$ is $\Theta(m^{p-k})$. Moreover $F$ is column maximal (a weak form of a boundary case), namely for any column $\alpha$ not in $F$ we have that forb$(m, [F|\alpha])$ is $\Omega(m^{p-k+1})$.*

# Induction

Let $A$ be an $m \times forb(m, F_7)$ simple matrix with no configuration $F_7$. We can select a row $r$ and reorder rows and columns to obtain

$$A = \begin{matrix} \text{row } r \\ \\ \end{matrix} \left[ \begin{matrix} 0 & \cdots & 0 & 1 & \cdots & 1 \\ B_r & & C_r & C_r & & D_r \end{matrix} \right].$$

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Now $[B_r C_r D_r]$ is an $(m-1)$-rowed simple matrix with no configuration $F_7$. Also $C_r$ is an $(m-1)$-rowed simple matrix with no configurations in $\mathcal{F}$ where $\mathcal{F}$ is derived from $F_7$.

$C_r$ has no $F$ in

$$\mathcal{F} = \left\{ \begin{bmatrix} 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix} \right\}$$

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Then

$\|A\| = forb(m, F_7) = \|B_r C_r D_r\| + \|C_r\| \leq forb(m-1, F_7) + \|C_r\|$.

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$\|A\| = forb(m, F_7) = \|B_r C_r D_r\| + \|C_r\| \le forb(m-1, F_7) + \|C_r\|$.

To show $forb(m, F_7)$ is quadratic it would suffice to show $\|C_r\|$ is linear for some choice of $r$.

# Repeated Induction

Let $C_r$ be an $(m-1)$-rowed simple matrix with no configuration in $\mathcal{F}$. We can select a row $s_i$ and reorder rows and columns to obtain

$$C_r = \begin{matrix}\text{row } s_i\end{matrix} \begin{bmatrix} 0 & \cdots & 0 & 1 & \cdots & 1 \\ E_i & & G_i & G_i & & H_i \end{bmatrix}.$$

To show $\|C_r\|$ is linear it would suffice to show $\|G_i\|$ is bounded by a constant for some choice of $s_i$. Our proof shows that assuming $\|G_i\| \geq 8$ for all choices $s_i$ results in a contradiction.

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$$C_r = \begin{array}{r} \text{row } s_i \\ \\ L_i\{ \end{array} \left[ \begin{array}{cccccc} 0 & \cdots & 0 & 1 & \cdots & 1 \\ E_i & & G_i & G_i & & H_i \\ \hline \multicolumn{6}{c}{\text{columns} \subseteq [\mathbf{0}|I]} \end{array} \right] \}L_i \quad .$$

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We may choose $s_1$ and form $L_1$.
Then choose $s_2 \in L_1$ and form $L_2$.
Then choose $s_3 \in L_2$ and form $L_3$.
etc.
We can show the sets $L_1 \backslash s_2, L_2 \backslash s_3, L_3 \backslash s_4, \ldots$ are disjoint.
Assuming $\|G_i\| \geq 8$ for all choices $s_i$ results in $|L_i \backslash s_{i+1}| \geq 3$ which yields a contradiction.

THANKS to the organizers, particularly Sali Attila!