Forbidden Configurations and Indicator Polynomials

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Introduction

The use of indicator polynomials was explored in a joint paper with Fleming, Füredi and Sali. This talk focuses on joint work with Balin Fleming that led to a breakthrough for Forbidden Configurations. Füredi and Sali continue to explore applications to critical hypergraphs.

Forbidden Configuration Survey at www.math.ubc.ca/∼anstee

Definition We say that a matrix A is *simple* if it is a (0,1)-matrix with no repeated columns. We can think of A as the incidence matrix of some set system $\mathcal{F}$.

$[m] = \{1, 2, \ldots, m\}$

Let S be a finite set.

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Some matrix notations are helpful:

K_k is the $k \times 2^k$ simple matrix $\approx 2^{[k]}$

Definition Given a matrix F, we say that A has F as a *configuration* if there is a submatrix of A which is a row and column permutation of F.

$$F = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix} \quad \in \quad A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 \end{bmatrix}$$

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We consider the property of forbidding a configuration F in A for which we say F is a *forbidden configuration* in A.

Definition Let $\text{forb}(m, F)$ be the largest function of m and F so that there exist a $m \times \text{forb}(m, F)$ simple matrix with *no* configuration F. Thus if A is any $m \times (\text{forb}(m, F) + 1)$ simple matrix then A contains F as a configuration.

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For example, $\quad \text{forb}(m, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}) = m + 1.$

Definition Let K_k denote the $k \times 2^k$ simple matrix of all possible columns on k rows (i.e. incidence matrix of $2^{[k]}$).

Theorem (Sauer 72, Perles and Shelah 72, Vapnik and Chervonenkis 71)

$$forb(m, K_k) = \binom{m}{k-1} + \binom{m}{k-2} + \cdots \binom{m}{0} = \Theta(m^{k-1})$$

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Order Shattered Sets

Let $\mathcal{F} \subseteq 2^{[m]}$. We say $S = \{i_1, i_2, i_3\}$ is order-shattered by $\mathcal{F}$ (or the associated incidence matrix A) if there are 2^3 columns of A

$$\begin{bmatrix} * & * & * & * & * & * & * & * \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ \delta & \delta & \epsilon & \epsilon & \kappa & \kappa & \lambda & \lambda \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ \beta & \beta & \beta & \beta & \gamma & \gamma & \gamma & \gamma \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ \alpha & \alpha & \alpha & \alpha & \alpha & \alpha & \alpha & \alpha \end{bmatrix} \begin{matrix} \\ \longrightarrow \text{row } i_1 \\ \\ \longrightarrow \text{row } i_2 \\ \\ \longrightarrow \text{row } i_3 \\ \\ \end{matrix}$$

Note that $A|_S$ has K_3. The symbols $\alpha, \beta, \ldots, \lambda$ are for vectors of appropriate length and $*$ refers to arbitrary entries.

Using the definition of order shattered sets we define

$$osh(\mathcal{F}) = \{S \in 2^{[m]} \,:\, S \text{ is order shattered by } \mathcal{F}\}$$

The set $osh(\mathcal{F})$ is a downset and moreover $|osh(\mathcal{F})| = |\mathcal{F}|$.

Theorem (A, Ronyai, Sali 02) The inclusion matrix $I(osh(\mathcal{F}), \mathcal{F})$ is nonsingular over every field.

Let A be an m-rowed simple matrix which has no configuration K_k. For any k-set of rows $S \in \binom{[m]}{k}$, we let $A|_S$ denote the submatrix of A given by the rows of S. Since A has no K_k, then for every k-set S of rows we have that $A|_S$ has an absent $k \times 1$ (0,1)-column.

Remark If A has the property that for every k-set of rows $S \in \binom{[m]}{k}$ we have that $A|_S$ has an absent column, then A has no K_k and so has at most $O(m^{k-1})$ columns.

Let B be a $k \times (k+1)$ simple matrix with one column of each column sum. For a matrix C, let $t \cdot C$ denote the matrix $[CCC \cdots C]$ from concatenating t copies of C. Let

$$F_B(t) = [K_k \ t \cdot [K_k \backslash B]]$$

Let k, t be given. Let A be any m-rowed simple matrix which has no configuration $F_B(t)$. Then for any k-set of rows $S \in \binom{[m]}{k}$, either $A|_S$ has an absent column or $A|_S$ has two columns which appear at most t times each.

We say such columns are in short supply.

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Idea: We wish to show that we could delete $O(m^{k-1})$ columns from A to obtain A' where A' has an absent column for each k-set of rows and hence A' has at most $O(m^{k-1})$ columns and so A has at most $O(m^{k-1})$ columns.

Assume A is an m-rowed simple matrix with no $F_B(t)$.

Let $S \in \binom{[m]}{k}$ and let α be a $k \times 1$ column which is in short supply on S.
We say that an $m \times 1$ column γ violates S (for the chosen α) if

$$\gamma|_S = \alpha.$$

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Let $\mathcal{T} \subseteq \binom{[m]}{k}$ be the set of k-sets S for which there are (at least) two $k \times 1$ columns α, β in short supply (no column absent).

We could eliminate $\leq t|\mathcal{T}|$ columns with violations on $S \in \mathcal{T}$ from A to obtain A' which has an absent column on each k-set of rows. Unfortunately $|\mathcal{T}|$ can be too large. We need a better way to estimate the number of columns in A that have violations on $S \in \mathcal{T}$.

Multilinear Indicator Polynomials

Let $S = \{i_1, i_2, \ldots, i_k\} \in \binom{[m]}{k}$
Let $x_1, x_2, \ldots, x_m$ be variables. Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)^T$ be a $k \times 1$ (0,1)-column. Let a be the number of 1's in α.

$$f_{S,\alpha}(\mathbf{x}) = \prod_{j=1}^{k} (x_{i_j} - \alpha_j)$$

For a $m \times 1$ (0,1)-column γ

$$f_{S,\alpha}(\bar{\gamma}) = \begin{cases} (-1)^a & \text{if } \gamma|_S = \alpha \\ 0 & \text{otherwise} \end{cases}$$

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$$f_{S,\alpha}(\mathbf{x}) = \prod_{j=1}^{k}(x_{i_j} - \alpha_j)$$

We check that degree of $f_{S,\alpha}(\mathbf{x})$ is k with leading term

$$\prod_{j=1}^{k} x_{i_j}$$

Assume $S \in \mathcal{T}$ and there are two $k \times 1$ columns α, β in short supply (no column absent) and the two indicator polynomials $f_{S,\alpha}$, $f_{S,\beta}$. We set

$$f_S(\mathbf{x}) = a_1 f_{S,\alpha}(\mathbf{x}) + a_2 f_{S,\beta}(\mathbf{x})$$

$$a_1 = +1, \qquad a_2 = -1$$

We have that for a $m \times 1$ (0,1)-column γ

$$f_S(\bar{\gamma}) \begin{cases} \neq 0 & \text{if } \gamma|_S = \alpha \text{ or } \gamma|_S = \beta \\ = 0 & \text{otherwise} \end{cases}$$

and degree of $f_S(\mathbf{x})$ is (at most) $k-1$ since the leading terms of degree k of $f_{S,\alpha}(\mathbf{x})$ and $f_{S,\beta}(\mathbf{x})$ will cancel.

Let $\mathcal{T} \subseteq \binom{[m]}{k}$ be the set of all k-tuples S for which two columns, say α, β are in short supply.

An independent set $\mathcal{I} = (S_i)$ is an ordered list $S_1, S_2, \ldots \in \mathcal{T}$ and $k \times 1$ (0,1)-columns $\gamma_1, \gamma_2, \ldots$ of A so that

γ_i violates S_i for two chosen columns but violates no S_j with $j < i$

An independent set can be found by a greedy approach.

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An independent set can be found by a greedy approach.

Theorem *If $\mathcal{I} = (S_i)$ is an independent set, then the indicator polynomials f_S are linearly independent.*

Proof: Form the matrix of order $|\mathcal{I}|$ with ij entry equal to

$$f_{S_j}(\bar{\gamma}_i)$$

The matrix is upper triangular with nonzeros on the diagonal.

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Theorem *If $\mathcal{I} = (S_i)$ is an independent set, and the indicator polynomials f_S are degree at most d then*

$$|\mathcal{I}| \leq \binom{m}{d} + \binom{m}{d-1} + \cdots + \binom{m}{0}$$

$$= \Theta(m^d).$$

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$$= \Theta(m^d).$$

In our case the indicator polynomials have degree $k-1$ and so $|\mathcal{I}|$ is $O(m^{k-1})$.

Theorem $\text{forb}(m, F_B(t))$ is $\Theta(m^{k-1})$.

Proof: Assume A is a matrix with no $F_B(t)$. For a given set $S \in \mathcal{T} \subseteq \binom{[m]}{k}$ there are at most $2t$ columns with violations of the two chosen columns in short supply on S. Let $\mathcal{I} = (S_i)$ be a maximal independent set with indicator polynomials f_{S_i}. Thus we may remove $2t|\mathcal{I}|$ or $O(m^{k-1})$ columns and remove all violations on the two chosen $k \times 1$ columns for each $S \in \mathcal{T}$ and so on each $S \in \mathcal{T}$ there will be an absent column. The resulting matrix has at most $O(m^{k-1})$ columns and so A has at most $O(m^{k-1})$ columns.

There is one more k-rowed configuration F, for each k, with forb(m, F) being $\Theta(m^{k-1})$. Let $k = 4$ and let D be the $k \times (11)$ simple matrix with all columns of sum at least 1 that do not simultaneously have 1's in rows 1 and 2. We take $F_D(t) = [\mathbf{0}_k \; (t+1) \cdot D]$ which for $k = 4$ becomes

$$F_D(t) = \left[\begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \end{array} \; (t+1) \cdot \left[\begin{array}{ccccccccccc} 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \end{array} \right] \right]$$

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If a matrix A has no $F_D(t)$ then each 4-set of rows $\{i_1, i_2, i_3, i_4\}$ in some ordering has one of the following occur:

	no
i_1	0
i_2	0
i_3	0
i_4	0

We have one more k-rowed configuration F, for each k, with $\mathrm{forb}(m,F)$ being $O(m^{k-1})$. Let D be the $k \times (2^k - 2^{k-2} - 1)$ simple matrix with all columns of sum at least 1 that do not simultaneously have 1's in rows 1 and 2. We take $F_D(t) = [\mathbf{0}_k \; (t+1)\cdot D]$ which for $k = 4$ becomes

$$F_D(t) = \left[\begin{array}{cc} 0 & \\ 0 & \\ 0 & (t+1)\cdot \\ 0 & \end{array}\left[\begin{array}{ccccccccccc} 0&0&0&1&1&1&1&0&0&0&0 \\ 0&0&0&0&0&0&0&1&1&1&1 \\ 1&0&1&0&1&0&1&0&1&0&1 \\ 0&1&1&0&0&1&1&0&0&1&1 \end{array}\right]\right]$$

If a matrix A has no F then each 4-set of rows $\{i_1, i_2, i_3, i_4\}$ in some ordering has one of the following occur:

	no	
i_1	0	
i_2	0	or at least two columns are in short supply
i_3	0	
i_4	0	

We have one more k-rowed configuration F, for each k, with forb(m,F) being $O(m^{k-1})$. Let $k=4$ and let D be the $k\times 11$ simple matrix with all columns of sum at least 1 that do not simultaneously have 1's in rows 1 and 2. We take $F_D(t)=[\mathbf{0}_4\,(t+1)\cdot D]$ which becomes

$$F_D(t)=\left[\begin{array}{c} 0\\0\\0\\0\end{array}\;(t+1)\cdot\left[\begin{array}{ccccccccccc} 0&0&0&1&1&1&1&0&0&0&0\\ 0&0&0&0&0&0&0&1&1&1&1\\ 1&0&1&0&1&0&1&0&1&0&1\\ 0&1&1&0&0&1&1&0&0&1&1\end{array}\right]\right]$$

If a matrix A has no F then each 4-set of rows $\{i_1,i_2,i_3,i_4\}$ in some ordering has one of the following occur:

$$\begin{array}{cccc} & \text{no} & & \leq t\\ i_1 & 0 & & 0\\ i_2 & 0 & \text{or at least two columns are in short supply or} & 0\\ i_3 & 0 & & 0\\ i_4 & 0 & & 1\end{array}.$$

The case of one column in short supply makes the proof much more difficult. We can find ways to eliminate $O(m^3)$ columns and make a column absent on many 4-sets of rows. But some are left.

A typical situation if we avoid $F_D(t)$ could be:

	$\leq t$	$\leq t$	$\leq t$	$\leq t$	$\leq t$	$\leq t$	no
i_1	0	0	1	0	1	0	
i_2	0	0	1	0			0
i_3	0	0			1	0	0
i_4	1		0	1	0	1	0
i_5		1	1	1	1	1	0

We only need a few of these columns to get our reduction in degree. Note that

	$\leq t$
i_1	1
i_2	
i_3	1
i_4	0
i_5	1

$\Rightarrow$

	$\leq t$	$\leq t$
i_1	1	1
i_2	1	0
i_3	1	1
i_4	0	0
i_5	1	1

$|S| = 5$

$$f_{S,\alpha}(\mathbf{x}) = \prod_{j=1}^{5}(x_{i_j} - \alpha_j)$$

For a 5×1 vector α, we check that degree of $f_{S,\alpha}(\mathbf{x})$ is 5 with leading term

$$\prod_{j=1}^{5} x_{i_j}$$

Consider $\sum y_i f_{S,\alpha(i)}(\mathbf{x})$ for some 5×1 columns $\alpha(1), \alpha(2), \ldots$. We can cancel the terms of degree 5 if $\mathbf{1}^T\mathbf{y} = 0$.

$|S| = 5$

$$f_{S,\alpha}(\mathbf{x}) = \prod_{j=1}^{5}(x_{i_j} - \alpha_j)$$

The terms of degree 4 in $f_{S,\alpha}(\mathbf{x})$ are

$$\alpha_r \prod_{j\in S\backslash r} x_j = \begin{cases} \prod_{j\in S\backslash r} x_j & \text{if } \alpha_r = 1 \\ 0 & \text{otherwise} \end{cases}$$

Consider $\sum y_i f_{S,\alpha(i)}(\mathbf{x})$ for some 5×1 columns $\alpha(1), \alpha(2), \ldots$. Let M denote the matrix whose columns are the vectors $\alpha(1), \alpha(2), \ldots$. We can cancel the terms of degree 4 if $M\mathbf{y} = 0$.

We are trying to find solutions to $M\mathbf{y} = \mathbf{0}$ with $\mathbf{1}^T\mathbf{y} = 0$. We are able to get two easy solutions to $M\mathbf{y} = \mathbf{0}$:

$\mathbf{y}$	-1	$+1$	-1	$+1$	-1	$+1$	-1
i_1	1	1	1	1	0	0	0
i_2	1	1	0	0	0	0	0
i_3	0	1	1	0	0	0	0
i_4	0	0	0	0	0	1	1
i_5	1	1	1	0	0	1	0

$\mathbf{y}$	$+1$	$+1$	-1
i_1	0	0	0
i_2	0	0	0
i_3	0	0	0
i_4	1	0	1
i_5	0	1	1

If we add the two solutions together, we obtain a solution $\mathbf{y}$ whose sum of coefficients is 0 i.e. $\mathbf{1}^T\mathbf{y} = 0$.

Adding the two previous solutions together we obtain a solution to $M\mathbf{y} = \mathbf{0}$ with $\mathbf{1}^T\mathbf{y} = 0$:

$\mathbf{y}$	-1	$+1$	-1	$+1$	-1	$+1$
i_1	1	1	1	1	0	0
i_2	1	1	0	0	0	0
i_3	0	1	1	0	0	0
i_4	0	0	0	0	0	0
i_5	1	1	1	0	0	1

We obtain an indicator polynomial $f(\mathbf{x})$ of degree 3 with $f(\bar{\gamma}) \neq 0$ if and only if γ violates one of the 6 listed columns on 5 rows. Note that $f(\mathbf{x})$ is an indicator polynomial for the 6 columns in short supply on the 5 rows but is not an indicator polynomial for all columns in short supply (or absent) on the 5 rows.

Theorem $\mathrm{forb}(m, F_D(t))$ is $O(m^3)$.

Proof: Assume A has no $F_D(t)$ and that we have deleted $O(m^3)$ columns. Consider the cases above for $S \in \binom{[m]}{5}$ and for which we have an indicator polynomial of degree at most 3 counting some violations (6 in example above).

Then we can create a maximal independent set $\mathcal{I} = (S_i)$ as before and given that the indicator polynomials are of degree at most 3, we can eliminate $O(m^3)$ columns. Further eliminations of $O(m^3)$ columns are required before there is guaranteed to be an absent column on each 4-set of $\binom{[m]}{4}$ at which point we conclude that $O(m^3)$ columns remain and so A has at most $O(m^3)$ columns.

Theorem *Let k, t be given positive integers with $k \geq 2$, $t \geq 1$. Let D be the $k \times (2^k - 2^{k-2} - 1)$ simple matrix with all columns of sum at least 1 that do not simultaneously have 1's in rows 1 and 2. We take $F_D(t) = [\mathbf{0}_k \, (t+1) \cdot D]$ Then*

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Theorem Let k be given and assume F is a k-rowed configuration which is not a configuration in $F_B(t)$ (for any choice of B as a $k \times (k+1)$ simple matrix with one column of each column sum and for any t) and not in $F_D(t)$ (for any t). Then forb(m, F) is $\Theta(m^k)$.

Where could we go from here?

Linear algebra does work for the following case for which we already had an alternate proof.

$$\text{no configuration } F = (t+1)\cdot\begin{bmatrix}1 & 1\\1 & 0\\0 & 1\\0 & 0\end{bmatrix} \quad \text{forces} \quad \begin{matrix}\le t & \le t & \le t & \le t\\1 & 0 & 0 & 1\\1 & 1 & 0 & 0\\0 & 1 & 1 & 0\\0 & 0 & 1 & 1\end{matrix}$$

on any 4 rows. We can form a degree 2 indicator polynomial

$$f(\mathbf{x}) = x_1x_2 - x_2x_3 + x_3x_4 - x_1x_4$$

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$$f(\mathbf{x}) = x_1x_2 - x_2x_3 + x_3x_4 - x_1x_4$$

Theorem Let $t \ge 1$ be given. Then $\mathrm{forb}(m, F)$ is $\Theta(m^2)$.

Now if A has no configuration $F = (t+1) \cdot \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}$

$$\text{this forces } \begin{array}{ccc} \le t & \le t & \le t \\ 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array} \quad \text{or} \quad \begin{array}{ccc} \le t & \le t & \le t \\ 0 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{array}$$

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Conjecture Let $t \geq 2$ be given. Then forb(m, F) is $\Theta(m^2)$.

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Thanks to the organizers Chris Godsil, Peter Sin and Qing Xiang!

Thanks to all the participants for a wonderful conference!

The building blocks of our constructions are I, I^c and T:

$$I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad I_4^c = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}, \quad T_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Note that

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} \notin I, \quad \begin{bmatrix} 0 \\ 0 \end{bmatrix} \notin I^c, \quad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \notin T$$

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Definition Given an $m_1 \times n_1$ matrix A and a $m_2 \times n_2$ matrix B we define the product $A \times B$ as the $(m_1 + m_2) \times (n_1 n_2)$ matrix consisting of all $n_1 n_2$ possible columns formed from placing a column of A on top of a column of B. If A, B are simple, then $A \times B$ is simple. (A, Griggs, Sali 97)

$$\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} \times \begin{bmatrix}1&1&1\\0&1&1\\0&0&1\end{bmatrix} = \begin{bmatrix}1&1&1&0&0&0&0&0&0\\0&0&0&1&1&1&0&0&0\\0&0&0&0&0&0&1&1&1\\1&1&1&1&1&1&1&1&1\\0&1&1&0&1&1&0&1&1\\0&0&1&0&0&1&0&0&1\end{bmatrix}$$

Given p simple matrices $A_1, A_2, \ldots, A_p$, each of size $m/p \times m/p$, the p-fold product $A_1 \times A_2 \times \cdots \times A_p$ is a simple matrix of size $m \times (m^p/p^p)$ i.e. $\Theta(m^p)$ columns.

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The Conjecture

Definition Let $x(F)$ denote the largest p such that there is a p-fold product which does not contain F as a configuration where the p-fold product is $A_1 \times A_2 \times \cdots \times A_p$ where each $A_i \in \{I_{m/p}, I^c_{m/p}, T_{m/p}\}$.

Thus $x(F) + 1$ is the smallest value of p such that F is a configuration in every p-fold product $A_1 \times A_2 \times \cdots \times A_p$ where each $A_i \in \{I_{m/p}, I^c_{m/p}, T_{m/p}\}$.

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The conjecture has been verified for $k \times l$ F where $k = 2$ (A, Griggs, Sali 97) and $k = 3$ (A, Sali 05) and $l = 2$ (A, Keevash 06) and for k-rowed F with bounds $\Theta(m^{k-1})$ or $\Theta(m^k)$ plus other cases.