# Forbidden Configurations: Progress on a Conjecture

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**Definition** We say that a matrix $A$ is *simple* if it is a (0,1)-matrix with no repeated columns.

i.e. if $A$ is $m$-rowed then $A$ is the incidence matrix of some family $\mathcal{A}$ of subsets of $[m] = \{1, 2, \ldots, m\}$.

$$A = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathcal{A} = \{\emptyset, \{1,2,4\}, \{1,4\}, \{1,2\}, \{1,2,3\}, \{1,3\}\}$$

**Definition** We define $\|A\|$ to be the number of columns in $A$.

$$\|A\| = 6$$

**Definition** Given a matrix $F$, we say that $A$ has $F$ as a *configuration* if there is a submatrix of $A$ which is a row and column permutation of $F$.

$$F = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix} \quad \in \quad A = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & \boxed{1 \quad 0 \quad 1} & & 1 & \boxed{0} \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & \boxed{1 \quad 1 \quad 0} & & 0 & \boxed{0} \end{bmatrix}$$

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We consider the property of forbidding a configuration $F$ in $A$.

**Definition** Let

*forb*$(m, F)$= $\max\{\|A\| : A$ $m$-rowed simple, no configuration $F\}$

Thus if $A$ is any $m \times (\mathit{forb}(m, F) + 1)$ simple matrix then $A$ contains the configuration $F$.

**Definition** Let $K_k$ denote the $k \times 2^k$ simple matrix of all possible columns on $k$ rows.

**Theorem** (Sauer 72, Perles and Shelah 72, Vapnik and Chervonenkis 71)

$$forb(m, K_k) = \binom{m}{k-1} + \binom{m}{k-2} + \cdots + \binom{m}{0} \textit{ which is } \Theta(m^{k-1}).$$

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**Corollary** *Let $F$ be a $k \times \ell$ simple matrix. Then $\mathit{forb}(m, F) = O(m^{k-1})$.*

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**Theorem** (Füredi 83). *Let $F$ be a $k \times \ell$ matrix. Then $forb(m, F) = O(m^k)$.*

# Critical Substructures

**Definition** A *critical substructure* of a configuration $F$ is a minimal configuration $F'$ contained in $F$ such that

$$forb(m, F') = forb(m, F).$$

A critical substructure has an associated construction avoiding it that yields a lower bound on $forb(m, F)$.
Some other argument provides the upper bound for $forb(m, F)$.
A consequence is that for a configuration $F''$ where $F'$ is contained in $F''$ and $F''$ is contained in $F$, we deduce that

$$forb(m, F') = forb(m, F'') = forb(m, F).$$

# Critical Substructures for $K_4$

$$K_4 = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Critical substructures are $\mathbf{1}_4$, $K_4^3$, $K_4^2$, $K_4^1$, $\mathbf{0}_4$, $2 \cdot \mathbf{1}_3$, $2 \cdot \mathbf{0}_3$.
Note that $forb(m, \mathbf{1}_4) = forb(m, K_4^3) = forb(m, K_4^2) = forb(m, K_4^1)$ $= forb(m, \mathbf{0}_4) = forb(m, 2 \cdot \mathbf{1}_3) = forb(m, 2 \cdot \mathbf{0}_3)$.

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Note that $\mathit{forb}(m,\mathbf{1}_4)=\mathit{forb}(m,K_4^3)=\mathit{forb}(m,K_4^2)=\mathit{forb}(m,K_4^1)$ $=\mathit{forb}(m,\mathbf{0}_4)=\mathit{forb}(m,2\cdot\mathbf{1}_3)=\mathit{forb}(m,2\cdot\mathbf{0}_3)$.

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$$K_4 = \left[\begin{array}{ccccccccccccccc|c|} 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & \color{red}{0} \\ 1 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & \color{red}{0} \\ 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & \color{red}{0} \\ 1 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & \color{red}{0} \end{array}\right]$$

Critical substructures are $\mathbf{1}_4$, $K_4^3$, $K_4^2$, $K_4^1$, $\color{red}{\mathbf{0}_4}$, $2 \cdot \mathbf{1}_3$, $2 \cdot \mathbf{0}_3$.
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# Critical Substructures for $K_k$?

Critical $k$-rowed substructures for $K_k$ on $k$ rows are $K_k^\ell$ for $0 \le \ell \le k$. On $k-1$ rows we conjecture that $2 \cdot \mathbf{1}_{k-1}$ and $2 \cdot \mathbf{0}_{k-1}$ are the only critical $k-1$-rowed substructures. Proofs of required base cases elude us although computer investigations suggest we are correct.

# We can extend $K_4$ and yet have the same bound

$[K_4|\mathbf{1}_2\mathbf{0}_2] =$

$$\left[\begin{array}{cccccccccccccccc|c} 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \end{array}\right]$$

**Theorem** (A., Meehan) For $m \geq 5$, we have $forb(m, [K_4|\mathbf{1}_2\mathbf{0}_2]) = forb(m, K_4)$.

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$forb(m, [K_4|\mathbf{1}_2\mathbf{0}_2]) = forb(m, K_4)$.
We expect in fact that we could add many copies of the column $\mathbf{1}_2\mathbf{0}_2$ and obtain the same bound, albeit for larger values of $m$.

# A Product Construction

The building blocks of our product constructions are $I$, $I^c$ and $T$:

$$I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad I_4^c = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}, \quad T_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

**Theorem** (Balogh, Bollobás 05) Let $k$ be given. Then $forb(m, \{I_k, I_k^c, T_k\})$ is $O(1)$.

**Definition** Given two matrices $A, B$, we define the product $A \times B$ as the matrix whose columns are obtained by placing a column of $A$ on top of a column of $B$ in all possible ways. (A, Griggs, Sali 97)

$$\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} \times \begin{bmatrix}1&1&1\\0&1&1\\0&0&1\end{bmatrix} = \begin{bmatrix}1&1&1&0&0&0&0&0&0\\0&0&0&1&1&1&0&0&0\\0&0&0&0&0&0&1&1&1\\1&1&1&1&1&1&1&1&1\\0&1&1&0&1&1&0&1&1\\0&0&1&0&0&1&0&0&1\end{bmatrix}$$

Given $p$ simple matrices $A_1, A_2, \ldots, A_p$, each of size $m/p \times m/p$, the $p$-fold product $A_1 \times A_2 \times \cdots \times A_p$ is a simple matrix of size $m \times (m/p)^p$ i.e. $\Theta(m^p)$ columns.

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# Examples

$$[01] \times [01] = K_2$$

$$\overbrace{[01] \times [01] \times \cdots \times [01]}^{k} = K_k$$

$I_{m/2} \times I_{m/2}$ is vertex-edge incidence matrix of $K_{m/2,m/2}$

# The Conjecture

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**Definition** Let $F$ be given. Let $x(F)$ denote the largest $p$ such that there is a $p$-fold product which does not contain $F$ as a configuration where the $p$-fold product is $A_1 \times A_2 \times \cdots \times A_p$ where each $A_i \in \{I_{m/p}, I^c_{m/p}, T_{m/p}\}$.

**Conjecture** (A, Sali 05) *forb*$(m, F)$ *is* $\Theta(m^{x(F)})$.

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The conjecture has been verified for $k \times \ell$ $F$ where $k = 2$ (A, Griggs, Sali 97) and $k = 3$ (A, Sali 05) and $l = 2$ (A, Keevash 06) and for $k$-rowed $F$ with bounds $\Theta(m^{k-1})$ or $\Theta(m^k)$ (A, Fleming 10) plus other cases.

In order for a 4-rowed $F$ to have $forb(m, F)$ be quadratic in $m$, the associated simple matrix must have a quadratic bound. Using a result of A and Fleming, there are three simple column-maximal 4-rowed $F$ for which $forb(m, F)$ is quadratic. Here is one example:

$$F_8 = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{bmatrix}$$

How can we repeat columns in $F_8$ and still have a quadratic bound? We note that repeating either the column of sum 1 or the column of sum 3 will result in a cubic lower bound. Thus we only consider taking multiple copies of the columns of sum 2.

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How can we repeat columns in $F_8$ and still have a quadratic bound? We note that repeating either the column of sum 1 or the column of sum 3 will result in a cubic lower bound. Thus we only consider taking multiple copies of the columns of sum 2. For a fixed $t$, let

$$F_8(t) = \left[\begin{matrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{matrix} \; t \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \\ 0 & 0 \end{bmatrix}\right]$$

$$F_8(t) = \begin{bmatrix} 1 & 0 & 1 & 0 & & t\cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \\ 0 & 0 \end{bmatrix} \\ 0 & 1 & 0 & 1 & & \\ 0 & 0 & 1 & 1 & & \\ 0 & 0 & 1 & 1 & & \end{bmatrix}$$

**Theorem** (A, Raggi, Sali 09) Let $t$ be given. Then $forb(m, F_8(t))$ is $O(m^2)$. Moreover $F_8(t)$ is a boundary case, namely for any column $\alpha$ not already present $t$ times in $F_8(t)$, then $forb(m, [F_8(t)|\alpha])$ is $\Omega(m^3)$.

The proof of the upper bound is currently a rather complicated induction with some directed graph arguments.

For each $\alpha$ there are $\Omega(m^3)$ product constructions avoiding $[F_8(t)|\alpha]$.

# $5 \times 6$ Simple Configuration with Quadratic bound

The Conjecture predicts nine 5-rowed simple matrices $F$ which are boundary cases, namely $forb(m, F)$ is predicted to be $O(m^2)$ and for any column $\alpha$ we have $forb(m, [F|\alpha])$ being $\Omega(m^3)$. Such $F$ happen all to be $5 \times 6$ simple matrices and we have handled the following case.

$$F_7 = \begin{bmatrix} 1 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

**Theorem** (A, Raggi, Sali) $forb(m, F_7)$ is $O(m^2)$.
The proof is currently a rather complicated induction.

# All 6-rowed Configurations with Quadratic Bounds

$$G_{6\times 3} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

**Theorem** (A,Raggi,Sali) Let $F$ be any 6-rowed configuration. Then $forb(m, F)$ is $O(m^2)$ if and only if $F$ is a configuration in $G_{6\times 3}$.

**Proof:** We use induction and the bound for $F_7$.

# Induction

Let $A$ be an $m \times forb(m, F_7)$ simple matrix with no configuration $F_7$. We can select a row $r$ and reorder rows and columns to obtain

$$A = \begin{matrix} \text{row } r \end{matrix} \left[ \begin{array}{cccccc} 0 & \cdots & 0 & 1 & \cdots & 1 \\ B_r & & C_r & C_r & & D_r \end{array} \right].$$

Now $[B_r C_r D_r]$ is an $(m-1)$-rowed simple matrix with no configuration $F_7$. Also $C_r$ is an $(m-1)$-rowed simple matrix with no configurations in $\mathcal{F}$ where $\mathcal{F}$ is derived from $F_7$.

Then

$\|A\| = forb(m, F_7) = \|B_r C_r D_r\| + \|C_r\| \leq forb(m-1, F_7) + \|C_r\|$.

To show $\|A\|$ is quadratic it would suffice to show $\|C_r\|$ is linear for some choice of $r$.

# Repeated Induction

Let $C_r$ be an $(m-1)$-rowed simple matrix with no configuration in $\mathcal{F}$. We can select a row $s$ and reorder rows and columns to obtain

$$C_r = \begin{matrix}\text{row } s \\ \\ \end{matrix}\begin{bmatrix} 0 & \cdots & 0 & 1 & \cdots & 1 \\ E_s & & G_s & G_s & & H_s \end{bmatrix}.$$

To show $\|C_r\|$ is linear it would suffice to show $\|G_s\|$ is bounded by a constant for some choice of $s$. Our proof shows that assuming $\|G_s\| \geq 8$ for all choices $s$ results in a contradiction.
This repeated induction is used to show that $forb(m, F_7)$ is $O(m^2)$.

# An unusual Bound

**Theorem** (A,Raggi,Sali) $forb(m, \{T_2 \times T_2, T_2 \times I_2, I_2 \times I_2\})$ is $\Theta(m^{3/2})$.

$$T_2 \times T_2 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix}, \; T_2 \times I_2 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix},$$

$$I_2 \times I_2 = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

# Induction

Let $A$ be an $m \times forb(m, \mathcal{F})$ simple matrix with no configuration in $\mathcal{F} = \{T_2 \times T_2, T_2 \times I_2, I_2 \times I_2\}$. We can select a row $r$ and reorder rows and columns to obtain

$$A = \begin{array}{c} \text{row } r \\ \\ \end{array} \left[ \begin{array}{cccccc} 0 & \cdots & 0 & 1 & \cdots & 1 \\ B_r & & C_r & C_r & & D_r \end{array} \right].$$

To show $\|A\|$ is $O(m^{3/2})$ it would suffice to show $\|C_r\|$ is $O(m^{1/2})$ for some choice of $r$. Our proof shows that assuming $\|C_r\| > 16m^{1/2}$ for all choices $r$ results in a contradiction.

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