Forbidden Configurations: Extensions to the Complete Object

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The Extremal Problem

A $(0,1)$-matrix is *simple* if it has no repeated columns.
$\|A\|$ will denote the number of columns of matrix A.

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Having fixed some family of matrices $\mathcal{F}$, called a forbidden family, we will define to be the set

$$\mathrm{Avoid}(m,\mathcal{F}) = \{A : A \text{ is } m\text{-rowed simple and } \; F \nprec A \;\; \forall\, F \in \mathcal{F}\}.$$

Consequently, we let

$$\mathrm{forb}(m,\mathcal{F}) = \max_{A \in \mathrm{Avoid}(m,\mathcal{F})} \|A\|.$$

(When $\mathcal{F} = \{F\}$, we will write $\mathrm{Avoid}(m,F)$ and $\mathrm{forb}(m,F)$.)

The Extremal Problem

As with any extremal problem, we search for *constructions* and *bounds*. Constructions A avoiding a certain object give *lower bounds* whereas *upper bounds* on $\mathrm{forb}(m,\mathcal{F})$ require new proofs.

Example Constructions which achieve the bound for the matrix on the right have the matrix on the left as a configuration.

$$\mathrm{forb}\left(m, \left[\begin{array}{cccc} \overbrace{1 \quad 1 \quad \cdots \quad 1}^{t} \\ 1 \quad 1 \quad \cdots \quad 1 \\ 0 \quad 0 \quad \cdots \quad 0 \end{array}\right]\right) \le \mathrm{forb}\left(m, \left[\begin{array}{cccc} \overbrace{1 \quad 1 \quad \cdots \quad 1}^{t} \\ 1 \quad 1 \quad \cdots \quad 1 \\ 0 \quad 0 \quad \cdots \quad 0 \\ 0 \quad 0 \quad \cdots \quad 0 \end{array}\right]\right).$$

Multiple Copies of a Configuration

If F is a $(0,1)$-matrix, then $t \cdot F$ will denote the matrix

$$[\overbrace{F\ F\ \cdots\ F}^{t \text{ copies}}].$$

As intuition suggests, there exists some integer M so that, whenever $m \geq M$,

$$\mathrm{forb}(m, (t+1) \cdot F) > \mathrm{forb}(m, t \cdot F).$$

Multiple Copies of a Configuration

Let F be given where F is $k \times \ell$. We split this into the cases where $\ell = 1$ and $\ell \geq 2$. The latter is easy:

Case 1: $\ell \geq 2$. Assume the contrary $\mathrm{forb}(m, (t+1) \cdot F) = \mathrm{forb}(m, t \cdot F)$ and so take an $m \times n$ matrix $A \in \mathrm{Avoid}(m, t \cdot F)$ with

$$n = \mathrm{forb}(m, t \cdot F) = \mathrm{forb}(m, (t+1) \cdot F)$$

and some $m \times 1$ column α not in A. Considering $A' = [A|\alpha]$, we have that $(t+1) \cdot F \prec A'$ on some $((t+1)\ell)$-set of columns of A' and since $\ell \geq 2$, we can take a $t\ell$-subset of these, not including α, on which $t \cdot F \prec A$, a contradiction.

Multiple Copies of a Configuration

Case 2: $\ell = 1$ we introduce the notation $\mathbf{1}_p\mathbf{0}_q$ to denote columns of p 1s on top of q 0s.

The following theorem of Keevash (2015) is useful for constructions:

Theorem Let p, λ be given. There exists some $A \in \mathrm{Avoid}(m, (\lambda+1)\cdot \mathbf{1}_p)$ whose column sums are all $p+1$ and $\|A\| = \frac{\lambda}{p+1}\binom{m}{p}$ for m, p, t satisfying $\binom{p+1-i}{p-i}$ divides $\binom{m-i}{p-i}$ for $i = 1, 2, \ldots, p-1$. ∎

When $p > q$, a result of Anstee, Barekat, and Pellegrin (2019) provides exact bounds, for large enough m, that grow with t. From the exact bounds it immediately follows that

$$\left|\mathrm{forb}(m, t\cdot \mathbf{1}_p\mathbf{0}_q) - \left(1 + \frac{t-2}{p+1}\right)\frac{m^p}{p!}\right| \le c_1 m^{p-1}.$$

When $p = q$, no exact bound is known but similar arguments apply.
∎

The bound on K_k

Let K_k be the $k \times 2^k$ matrix of all possible columns on k rows. The following, due to Sauer 72, Perles, and Shelah 72, and Vapnik and Chervonenkis 71, is a central result in forbidden configurations.

Theorem

$$\mathrm{forb}(m, K_k) = \binom{m}{0} + \binom{m}{1} + \binom{m}{2} + \cdots + \binom{m}{k-1}. \qquad \blacksquare$$

First use Pascal's identities to arrange the above expansions as

$$\begin{aligned}\binom{m-1}{0} + \binom{m-1}{1} + \binom{m-1}{2} + \cdots + \binom{m-1}{k-1} \\ + \binom{m-1}{0} + \binom{m-1}{1} + \cdots + \binom{m-1}{k-2}\end{aligned}$$

yielding $\mathrm{forb}(m-1, K_k) + \mathrm{forb}(m-1, K_{k-1}) = \mathrm{forb}(m, K_k)$.

The bound on K_k

Let us prove the bound by illustrating the method of standard induction.

Given a matrix A on m rows avoiding K_k, we can permute the rows and columns of A as

$$\text{row } r \rightarrow \left[\begin{array}{cccc} 1\,1\,\cdots\,1 & 1\,1\,\cdots\,1 & 0\,0\,\cdots\,0 & 0\,0\,\cdots\,0 \\ B_r & C_r & C_r & D_r \end{array}\right]$$

where C_r are those columns which would be repeated upon the deletion of row r. The matrices C_r and $[B_r\ C_r\ D_r]$ are $(m-1)$-rowed simple. $[B_r\ C_r\ D_r]$ has no K_k but we can say *more* about C_r: since C_r appears under 1s and 0s, C_r has no K_{k-1}. Therefore, with $\|A\| = \|[B_r\ C_r\ D_r]\| + \|C_r\|$,

$$\|A\| \leq \mathrm{forb}(m-1, K_k) + \mathrm{forb}(m-1, K_{k-1}) = \mathrm{forb}(m, K_k),$$

precisely the inductive result we require. ■

Our main question is for which B is it true that

$$\mathrm{forb}(m, [K_4|B]) = \mathrm{forb}(m, K_4)$$

(at least for large m)? We make progress.

The bound on K_k

Matrices A on m rows with $\|A\| = \mathrm{forb}(m, K_k)$ vary a great deal. They are **not** canonical.

The first 5×16 matrix has no K_3 because it has no submatrix $[1\ 0\ 1]^T$. The second is more random.

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \end{bmatrix}$$

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$$
\begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
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0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0
\end{bmatrix}
$$

What is missing?

no	no	no	no	no	no	no	no	no	no
1	1	1	1	1	1				
0	0	0				1	1	1	
1			1	1		1	1		1
	1		1		0	1		0	0
		1		1	1		1	1	1

Critical Substructures for K_4

A critical substructure of a configuration F is a minimal configuration $F' \prec F$ so that $\mathrm{forb}(m, F') = \mathrm{forb}(m, F)$.

$$K_4 = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Critical substructures are $\mathbf{1}_4$, K_4^3, K_4^2, K_4^1, $\mathbf{0}_4$, $2 \cdot \mathbf{1}_3$, $2 \cdot \mathbf{0}_3$.

Note that

$$\begin{aligned} \mathrm{forb}(m, \mathbf{1}_4) &= \mathrm{forb}(m, K_4^3) = \mathrm{forb}(m, K_4^2) = \mathrm{forb}(m, K_4^1) \\ &= \mathrm{forb}(m, \mathbf{0}_4) = \mathrm{forb}(m, 2 \cdot \mathbf{1}_3) = \mathrm{forb}(m, 2 \cdot \mathbf{0}_3). \end{aligned}$$

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Motivations

Can we add columns to K_4 and preserve its bound? The added columns must have column sum 2.

$$\left[\begin{array}{cccccccccccccccc|c} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \end{array}\right]$$

The 3×3 block $3 \cdot \mathbf{1}_3$ has a bound of $\binom{m}{0} + \binom{m}{1} + \binom{m}{2} + \frac{5}{4}\binom{m}{3}$, bigger than that of K_4.

Results in Anstee, Meehan (2011) state that

$$\mathrm{forb}(m, [K_4|\mathbf{1}_2\mathbf{0}_2]) = \mathrm{forb}(m, K_4)$$

for m large enough (actually $m \geq 5$). Generalizations were hindered in searching for base cases m in the standard induction. Using several stability lemmas, we can overcome these difficulties.

Product Construction

If F is $k_1 \times \ell_1$ and G is $k_2 \times \ell_2$, we will denote by $F \times G$ the $(k_1 + k_2) \times \ell_1\ell_2$ matrix consisting of every column of F appearing over every column of G.

In this way,

$$K_k = \overbrace{[1\ 0] \times [1\ 0] \times \cdots \times [1\ 0]}^{k \text{ times}}.$$

Main Theorems

Let

$$K_2^T = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad F_1 = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}.$$

Theorem Assume $k \geq 4$ and $t \geq 1$. There exists an m_k so that, for $m > m_k$, we have

$$\mathrm{forb}(m, [K_k | t \cdot (K_2^T \times K_{k-4})] = \mathrm{forb}(m, K_k). \qquad \blacksquare$$

The neat fact due to Gronau (1980) that $\mathrm{forb}(m, 2 \cdot K_k) = \mathrm{forb}(m, K_{k+1})$ is instrumental in proving:

Theorem Assume $k \geq 3$ and $t \geq 1$. There exists an m_k so that, for $m > m_k$, we have

$$\mathrm{forb}(m, [2 \cdot K_k | t \cdot (F_1 \times K_{k-3})]) = \mathrm{forb}(m, 2 \cdot K_k). \qquad \blacksquare$$

Proof of Theorem

$$F_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad F_3 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad F_4 = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Let $\mathcal{F} = \{[K_3|t \cdot F_2], [K_3|t \cdot F_3], [K_3|t \cdot F_4]\}$

$$\begin{array}{ccccc}
\textbf{Claim 1}(3) & & \textbf{Claim 2}(3) & & \textbf{Claim 3}(3) \\
\mathrm{forb}(m, \mathcal{F}) & \rightarrow & \mathrm{forb}(m, \mathcal{F}) & \rightarrow & A \in \mathrm{Avoid}(m, \mathcal{F}) \\
\leq \mathrm{forb}(m, K_3) + c & & = \mathrm{forb}(m, K_3) & & t \cdot F_2, t \cdot F_3, t \cdot F_4 \not\prec A|_S, \\
 & & & & \|A\| \leq forb(m, K_3) - m + 4t \\
 & \swarrow & & \swarrow & \downarrow \\
\textbf{Claim 1}(4) & & \textbf{Claim 2}(4) & & \textbf{Claim 3}(4) \\
\mathrm{forb}(m, [K_4|t \cdot K_2^T]) & \rightarrow & \mathrm{forb}(m, [K_4|t \cdot K_2^T]) & \rightarrow & A \in \mathrm{Avoid}(m, [K_4|t \cdot K_2^T]) \\
\leq \mathrm{forb}(m, K_4) + c_4 & & = \mathrm{forb}(m, K_4) & & t \cdot K_2^T \not\prec A|_S \\
 & & & & \|A\| \leq forb(m, K_4) - m + 4t \\
 & \swarrow & & \swarrow & \downarrow \\
\textbf{Claim 1}(5) & & \textbf{Claim 2}(5) & & \textbf{Claim 3}(5) \\
\mathrm{forb}(m, [K_5|t \cdot [01] \times K_2^T]) & \rightarrow & \mathrm{forb}(m, [K_5|t \cdot [01] \times K_2^T]) & \rightarrow & A \in \mathrm{Avoid}(m, [K_5|t \cdot [01] \times K_2^T]) \\
\leq \mathrm{forb}(m, K_5) + c_4 & & = \mathrm{forb}(m, K_5) & & t \cdot [01] \times K_2^T \not\prec A|_S \\
 & & & & \|A\| \leq forb(m, K_5) - m + 4t \\
\vdots & & \vdots & & \vdots
\end{array}$$

Proof of Theorem (outline)

To understand the somewhat complicated induction, consider the proof of **Claim 2**(4) that $\text{forb}(m, [K_4 | t \cdot K_2^T]) = \text{forb}(m, K_4)$ for m large.

We use **Claim 1**(4) and **Claim 3**(3) as well as some analysis of our standard induction.

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We use **Claim 1**(4) and **Claim 3**(3) as well as some analysis of our standard induction.
Let $A \in \mathrm{Avoid}(m, [K_4|t \cdot K_2^T])$. If $K_4 \nprec A$, then $\|A\| \leq \mathrm{forb}(m, K_4)$ as desired. So assume for some set of rows S, $K_4 \prec A|_S$.
Then $t \cdot K_2^T \nprec A|_S$ (actually $(t+1) \cdot K_2^T \nprec A|_S$ but who's counting).
Using standard induction we deduce that $t \cdot F_1$, $t \cdot F_2$, $t \cdot F_3 \nprec C_r|_{S \backslash r}$.

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We use **Claim 1**(4) and **Claim 3**(3) as well as some analysis of our standard induction.

Let $A \in \text{Avoid}(m, [K_4 | t \cdot K_2^T])$. If $K_4 \not\prec A$, then $\|A\| \leq \text{forb}(m, K_4)$ as desired. So assume for some set of rows S, $K_4 \prec A|_S$.

Then $t \cdot K_2^T \not\prec A|_S$ (actually $(t+1) \cdot K_2^T \not\prec A|_S$ but who's counting).

Using standard induction we deduce that $t \cdot F_1$, $t \cdot F_2$, $t \cdot F_3 \not\prec C_r|_{S \setminus r}$.

By **Claim 3**(3), we have $\|C_r|_{S \setminus r}\| \leq \text{forb}(m-1, K_3) - m + 4t$.

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Also by standard induction, $[B_r C_r D_r] \in \mathrm{Avoid}(m-1, [K_4|t \cdot K_2^T])$. Apply **Claim 1**(4) to obtain $\|[B_r C_r D_r]\| \leq \mathrm{forb}(m-1, K_4) + c_4$.

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Using the recursion $\mathrm{forb}(m-1, K_4) + \mathrm{forb}(m-1, K_3) = \mathrm{forb}(m, K_4)$ we obtain the result assuming $m > c_4 + 4t$. ∎

Problems

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Can we improve our result for K_4 (i.e. add more columns and get the same exact bound) or are the current results best possible (some constructions would be required)?

The following theorem indicates that we will certainly see a change in the bound of K_4 if we were to extend to $[K_4|K_4^2]$.

Theorem (Anstee, Fleming 2010) Let k be given and let B be an $k \times (k+1)$ matrix with one column of each column sum. Then $\mathrm{forb}(m, [K_k|t \cdot (K_k \backslash B)])$ is $\Theta(m^{k-1})$. Also if F is a k-rowed configuration and $K_k \prec F$, then $\mathrm{forb}(m, F)$ is $\Theta(m^{k-1})$ if and only if there is a t and $k \times (k+1)$ matrix B with one column of each column sum where $F \prec [K_k|t \cdot (K_k \backslash B)]$. ■

Problems

Let

$$F_5 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad F_6 = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}, \quad F_7 = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix}.$$

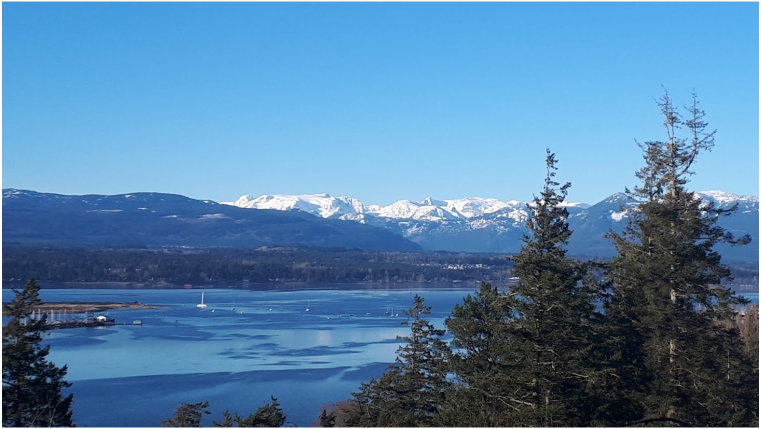
Problem Show that

$$\mathrm{forb}(m, [K_4|F_5]) > \mathrm{forb}(m, K_4) \quad \text{and} \quad \mathrm{forb}(m, [K_4|F_6]) > \mathrm{forb}(m, K_4).$$

■

Constructions are hard to come by. It is possible that even $\mathrm{forb}(m, [K_4|t \cdot F_7]) = \mathrm{forb}(m, K_4)$. We need some new constructions!

Thank You



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