Stochastic Equations of Super-Lévy Process with General Branching Mechanism

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Introduction

Let $\{X_t : t \geq 0\}$ be a binary branching super-Brownian motion (SBM). Then $X_t(dx) = X_t(x)dx$ and the density is the unique positive weak solution to (Konno-Shiga (1988) and Reimers (1989)):

$$\frac{\partial X_t(x)}{\partial t} = \frac{1}{2}\Delta X_t(x) + \sqrt{X_t(x)}\dot{W}_t(x), \quad t \geq 0, \quad x \in \mathbb{R}, \tag{1}$$

where $\dot{W}_t(x)$ is the derivative of a space-time Gaussian white noise (GWN).

- The pathwise uniqueness for (1) is unknown.
  Progress: Perkins, Sturm, Mytnik, etc.
- Xiong (2012) studied the pathwise uniqueness to SPDE for the distribution function process of the SBM.
  Pathwise uniqueness to similar equation see Dawson and Li (2012).
- This talk is to generalize the result of Xiong (2012) to the super-Lévy process with general branching mechanism.

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- $D(\mathbb{R}) := \{f : f$ is bounded right continuous increasing and $f(-\infty) = 0\}$.
  $M(\mathbb{R}) := \{$finite Borel measures on $\mathbb{R}\}$.
  There is a 1-1 correspondence between $D(\mathbb{R})$ and $M(\mathbb{R})$ assigning a measure to its distribution function. We endow $D(\mathbb{R})$ with the topology induced by this correspondence from the weak convergence topology of $M(\mathbb{R})$.

- The branching mechanism ϕ:

$$\phi(\lambda) = b\lambda + c\lambda^2/2 + \int_0^\infty (e^{-z\lambda} - 1 + z\lambda)m(dz).$$

- $M(\mathbb{R})$-valued $\{X_t\}$ process is called a super-Lévy process if

$$\begin{cases} \mathbf{E}_\mu\Big\{\exp[-\langle X_t, f\rangle]\Big\} = \exp\{-\langle \mu, v_t\rangle\}, \\ \frac{\partial}{\partial t} v_t(x) = Av_t(x) + \phi(v_t(x)), \quad v_0(x) = f(x). \end{cases}$$

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Our aim in this talk is that under a mild condition on A, $\{Y_t\}$, defined by $Y_t(x) = X_t(-\infty, x]$, is the pathwise unique solution to

$$\begin{aligned} Y_t(x) &= Y_0(x) - b\int_0^t Y_s(x)ds + \sqrt{c}\int_0^t \int_0^{Y_s(x)} W(ds, du) \\ &\quad + \int_0^t \int_0^\infty \int_0^{Y_{s-}(x)} z\tilde{N}_0(ds, dz, du) + \int_0^t A^* Y_s(x)ds, \end{aligned} \tag{2}$$

where $W(ds, du)$ is a GWN and $\tilde{N}_0(ds, dz, du)$ compensated Poisson random measure (CPRM), A^* denotes the dual operator of A.

- Xiong (2012): $A = \Delta/2$ and $b = \tilde{N}_0 = 0$.
- Key approach: connecting (2) with a backward doubly SDE. Xiong (2012) used an L^2-argument. We use an L^1-argument.
- For $M(\mathbb{R})$-valued process $\{X_t\}$, its distribution $\{Y_t\}$ is $D(\mathbb{R})$-valued.

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Main results

Theorem 1

$D(\mathbb{R})$-valued process $\{Y_t\}$ is the distribution of a super-Lévy process iff there is, on an enlarged probability space, a GWN $\{W(ds,du)\}$ and a CPRM $\{\tilde{N}_0(ds,dz,du)\}$ so that $\{Y_t\}$ solves (2).

Let $(P_t)_{t\geq 0}$ be the transition semigroup of a Lévy process with generator A.

Condition 1

For some continuous function $(t,z)\mapsto p_t(z)$, $\alpha\in(0,1)$ and $C\in B[0,\infty)$,

$$P_t(x,dy)=p_t(y-x)dy \quad \text{and} \quad p_t(x)\leq t^{-\alpha}C(t), \qquad t>0,\ x,y\in\mathbb{R}.$$

The condition holds if A is the generator of a stable process with index in $(1,2]$.

Theorem 2

Under Condition 1, the pathwise uniqueness holds for (2) with $Y_0\in D(\mathbb{R})$.

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Theorem 2

Under Condition 1, the pathwise uniqueness holds for (2) with $Y_0 \in D(\mathbb{R})$.

Proof of Theorem 2

- Define ξ by

$$\xi(t) = \beta t + \sigma B_t + \int_0^t \int_{\{|z|\leq 1\}} z\tilde{M}(ds,dz) + \int_0^t \int_{\{|z|>1\}} zM(ds,dz) \qquad (3)$$

and independent of $\{W(ds,du)\}$ and $\{\tilde{N}_0(ds,dz,du)\}$ and $\xi_t^r = \xi(r\wedge t) - \xi(t)$.

- Take $T > 0$ and define GWN $W^T(ds,dx)$ and CPRM $\tilde{N}_0^T(ds,dz,du)$ by

$$W^T((0,t]\times A) = W([T-t,T)\times A), \quad \tilde{N}_0^T((0,t]\times B) = \tilde{N}_0([T-t,T)\times B).$$

From (2),

$$\begin{aligned} Y_{T-t}(x) &= Y_0(x) + \int_t^T A^* Y_{T-s}(x)ds + \sqrt{c}\int_{t-}^{T-}\int_0^{Y_{T-s}(x)} W_T(\overleftarrow{ds},du) \\ &\quad - \int_t^T bY_{T-s}(x)ds + \int_{t-}^{T-}\int_0^\infty\int_0^{Y_{(T-s)-}(x)} z\tilde{N}_T(\overleftarrow{ds},dz,du). \qquad (4) \end{aligned}$$

$W_T(\overleftarrow{ds},du)$ is the backward Itô's integral, i.e., in the Riemann sum approximating the stochastic integral, taking right end-points instead of the left ones.

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$W_T(\overleftarrow{ds}, du)$ is the backward Itô's integral, i.e., in the Riemann sum approximating the stochastic integral, taking right end-points instead of the left ones.

From (3) and (4), under Condition 1 for all $x \in \mathbb{R}$ and $0 \le r \le t \le T$ we have a.s.

$$\begin{aligned} Y_{T-t}(\xi_t^r + x) &= Y_0(\xi_T^r + x) - b \int_t^T Y_{T-s}(\xi_s^r + x)ds + \sigma \int_t^T \nabla Y_{T-s}(\xi_s^r + x)dB_s \\ &\quad + \sqrt{c} \int_{t-}^{T-} \int_0^{Y_{T-t}(\xi_s^r + x)} W_T(\overleftarrow{ds}, du) \\ &\quad + \int_{t-}^{T-} \int_0^{\infty} \int_0^{Y_{(T-s)-}(\xi_s^r + x)} z\tilde{N}_T(\overleftarrow{ds}, dz, du) \\ &\quad - \int_t^T \int_{\mathbb{R}^\circ} [Y_{T-s}(\xi_{s-}^r + x - z) - Y_{T-s}(\xi_{s-}^r + x)]\tilde{M}(ds, dz). \end{aligned} \quad (5)$$

Remark:

(i) The fourth and fifth terms are time-reversed martingales.

(ii) We cannot establish (5) simultaneously for all $(t, x) \in [r, T] \times \mathbb{R}$.

$t \mapsto Y_{T-t}(\xi_s^r + x)$ is neither right continuous nor left continuous.

(iii) The process defined by above general kind of SDE is unique.

(iv) Prove a generalized Itô's formula, which is initiated by Pardoux and Peng (1994).

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Further result: SPDE driven by α-stable noise

The weak solution for the following SPDE was constructed by Mytnik (2002):

$$\frac{\partial X_t(x)}{\partial t} = \frac{1}{2}\Delta X_t(x) + X_{t-}(x)^{\beta}\dot{L}, \qquad X_0 \geq 0,\ x \in \mathbb{R}^d, \tag{6}$$

where $L(ds, dx)$ is a one-sided, α-stable white noise without negative jumps, $1 < \alpha < \min(2, (2/d) + 1),\ \beta > 0,\ p := \alpha\beta < (2/d) + 1$.

- $p = 1$, the solution is a superprocess and the weak uniqueness holds.
- $p \neq 1$, the uniqueness for (6) and the properties of solution are unknown.
- We consider the case $d = 1$ and $p \in (0, \alpha)$ here.
  Other cases are being considered.

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Further result: SPDE driven by α-stable noise

The weak solution for the following SPDE was constructed by Mytnik (2002):

$$\frac{\partial X_t(x)}{\partial t} = \frac{1}{2}\Delta X_t(x) + X_{t-}(x)^{\beta}\dot{L}, \qquad X_0 \geq 0,\ x \in \mathbb{R}^d, \tag{6}$$

where $L(ds, dx)$ is a one-sided, α-stable white noise without negative jumps, $1 < \alpha < \min(2, (2/d) + 1),\ \beta > 0,\ p := \alpha\beta < (2/d) + 1$.

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• Equation (6) means:

$$\langle X_t, f\rangle = \langle X_0, f\rangle + \frac{1}{2}\int_0^t \langle X_s, f''\rangle ds + \int_0^t \int_{\mathbb{R}} X_{s-}(x)^\beta f(x) L(ds, dx). \tag{7}$$

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where $\tilde{N}_0(ds, dz, du, dv)$ is a CPRM.

• Similar to Theorem 1.1 (a) and 1.3 (a) in Mytnik and Perkins (2003) we have:
$X_t(\cdot)$ has a continuous version for fixed t.
Occupation density $\mathscr{Y}_t(x) := \int_0^t X_s(x) ds$ has a jointly continuous version.

• Connecting (8) with a backward doubly SDE, (8) has a pathwise uniqueness solution, which implies the weak uniqueness to (7).

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Thanks!

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