Lecture of 2012-06-14 **scribe: B. Kolesnik**

1 Last passage percolation trees vs. random independent trees

The lecture began by looking at computer simulation output comparing the $\mathbb{Z}^2_+$-LPP tree and the random independent tree (as discussed in the previous lecture). See Figure 1. Recall that the independent tree is formed by independently selecting an edge from each corner with selection probably biased towards the farthest of the two axes.

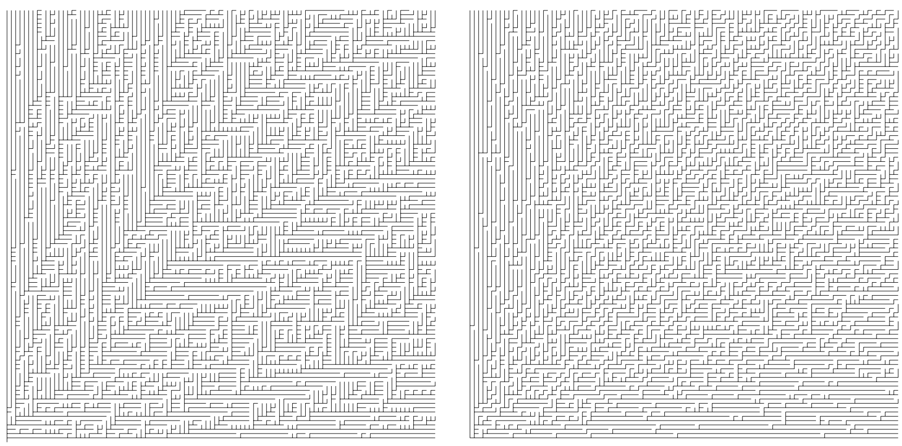

Figure 1: *At left:* Last passage percolation tree. *At right:* Random independent tree.

There are appreciable differences between the two trees. For instance, in the LPP tree there are several long stretches of vertical edges above the line $y = x$, and similarly many long stretches of horizontal edges below the line $y = x$. This makes sense, since if there is a heavy vertex below the line $y = x$ in the tree somewhere it is very likely that LPP paths will pass through this vertex and continue in the horizontal direction for some time (and vice versa for heavy vertices above the line $y = x$.). On the other hand, the random independent tree is more homogenous.

Note that any path in the LPP tree connects to the origin either through the point $z_1 = (0, 0) + e_1$ or the point $z_2 = (0, 0) + e_2$. We looked at computer output depicting the interface between the subtree through z_1 (coloured black) and the subtree through z_2 (coloured red). See Figure 2. Note the blue line is a path from the origin to (n, n). The interface between the red and black trees corresponds to an infinite path in the dual LPP tree. However, since the x-axis and y-axis are infinite paths in the LPP tree, the existence of this infinite path in the dual does not imply the existence of a non-trivial infinite path in the LPP tree. If we can show that there exists another infinite path in the dual tree then we will have shown that there is a non-trivial path in the LPP tree.

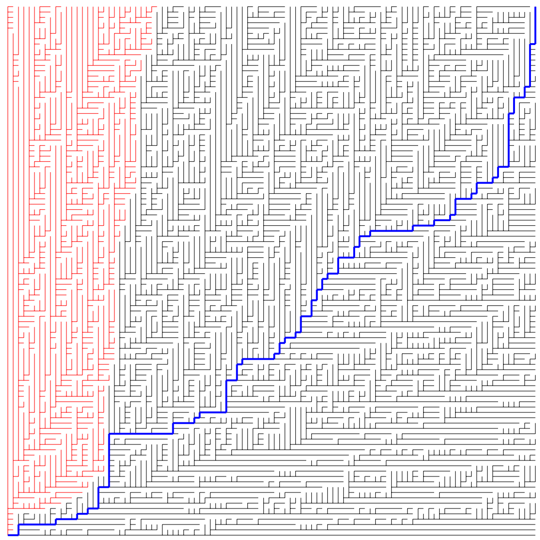

Figure 2: Dual path interface.

2 Last passage paths in $\mathbb{Z}_+^2$

The purpose of this section is to investigate the infinite geodesic paths in the $\mathbb{Z}_+^2$-LPP tree.

First of all, we examine LPP paths from 0 to a point $x \in \mathbb{N}^2$.

2.1 The last passage path from 0 to x is approximately straight

Our goal in this subsection is to show that the LPP path from 0 to x (for $|x|$ sufficiently large) is approximately a straight line. The general idea is to show that for any point y far away from the straight line from 0 to x, the LPP path from 0 to x passes through y with small probability. The essential ingredients in what follows are the super-additivity of G and the concavity of g, the limit shape of G.

Recall from previous lectures that $G(u, v)$ denotes the maximum weight over paths from u to v (since exponential weights are assigned, this path is unique a.s.). Simultaneously for all x,

$$G(0, nx)/n \xrightarrow{\text{a.s.}} g(x) \tag{1}$$

where

$$g(x) = (\sqrt{x}_1 + \sqrt{x}_2)^2.$$

In what follows, let '$0 \to y \to x$' denote the event that the LPP path from 0 to x passes through the point y. As discussed above, our aim is to show that $\mathbf{P}(0 \to y \to x)$ is small. From previous lectures, we know that G is superadditive:

$$G(0, x) \geq G(0, y) + G(y, x). \tag{2}$$

Note that the event $0 \to y \to x$ occurs if and only if equality is attained in (2).

g is a concave function. As can easily be checked, the Hessian of g has eigenvalues 0 and $-(x^2 + y^2)/2(xy)^{3/2} < 0$. Hence, for $y < x$,

$$g(x) \geq g(y) + g(x - y). \tag{3}$$

Furthermore, if $y = (\alpha x_1, \beta x_2)$ for some $0 < \alpha \neq \beta \leq 1$; since $1 > \sqrt{\alpha\beta} + \sqrt{(1-\alpha)(1-\beta)}$ unless $\alpha = \beta < 1$, we have that equality holds in (3) if and only if x and y are colinear.

By translation invariance,

$$G(y, x) \approx g(x - y).$$

Thus, in light of (2) and (3) and the discussion following it, we look into the accuracy of the approximation $G(x) = G(0, x) \approx g(x)$.

Recall Talagrand's inequality:

$$\mathbf{P}(|G(x) - \mathbf{E}G(x)| > s) \leq \exp(-cs^2/|x|). \tag{4}$$

Inequality (4) will prove useful for us, however, since $\mathbf{E}G(x) \neq g(x)$ its application will not be direct. Before proceeding, we establish an elementary result.

Lemma 1. *Suppose $Z_1, Z_2, \ldots, Z_n =_d Exp(\beta)$. Then*

$$\mathbf{E}\left(\max_{1\leq i\leq n} Z_i\right) \lesssim \log n/\beta$$

Proof. We have

$$\begin{aligned}
\mathbf{E}\left(\max_{1\leq i\leq n} Z_i\right) &\leq \log n/\beta + n\int_0^\infty \mathbf{P}(Z > \log n/\beta + t)dt \\
&\leq \log n/\beta + \int_0^\infty \exp(-\beta t)dt \\
&= (1 + \log n)/\beta.
\end{aligned}$$

The result follows. □

We are ready to prove our main lemma.

Lemma 2. *For sufficiently large $|x|$, there is a constant C such that*

$$0 \leq g(x) - \mathbf{E}G(x) \leq C\sqrt{|x|}\log|x|.$$

Proof. To get the first inequality, note that by super-additivity and translation invariance

$$\mathbf{E}G(nx) \geq n\mathbf{E}G(x).$$

Hence, dividing by n, taking limits and applying (1), we obtain

$$g(x) \geq \mathbf{E}G(x).$$

Before we argue for the second inequality, we prove a preliminary claim.

Claim 3. *For sufficiently large $|x|$, there exists a constant C_1 such that*

$$\mathbf{E}(G(2x) - 2G(x)) \leq C_1\sqrt{|x|}\log|x|.$$

Proof. By L denote the line with slope -1 and passing through the point x. See Figure 3. Note that the LPP path from 0 to $2x$ necessarily passes through L; and thus

$$G(2x) = \max_{y\in L}(G(0,y) + G(y,2x)). \tag{5}$$

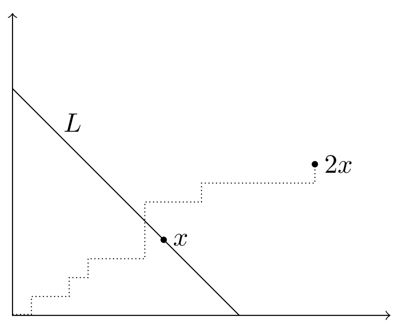

Figure 3: The line L corresponding to x

We apply (4) to y:

$$\mathbf{P}(|G(y) - \mathbf{E}G(y)| > s) \leq \exp(-cs^2/|y|) \leq \exp(-cs^2/|x|).$$

We see that each $|G(y) - \mathbf{E}G(y)|$ has an exponential tail with decay rate $\approx 1/\sqrt{|x|}$. Applying Lemma 1 we find that

$$\mathbf{E}\left(\max_{y\in L}|G(y) - \mathbf{E}G(y)|\right) \lesssim \sqrt{|x|}\log|x|. \tag{6}$$

Note that

$$G(y) \leq \mathbf{E}G(y) + |G(y) - \mathbf{E}G(y)|.$$

Hence

$$G(y) + G(y,2x) \leq \mathbf{E}G(y) + G(y,2x) + |G(y) - \mathbf{E}G(y)|$$

So by (5), (6) and the approximate concavity of G, there is a constant, C_1 say, so that

$$\begin{aligned}\mathbf{E}G(2x) &\leq \max_{y\in L}\mathbf{E}(G(0,y)+G(y,2x))+C_1\sqrt{|x|}\log|x| \\ &= 2\cdot\mathbf{E}G(0,x)+C_1\sqrt{|x|}\log|x|\end{aligned}$$

The claim follows. ☐

Suppose now the second inequality is invalid . Then for arbitrarily large K,

$$\mathbf{E}G(x) < g(x) - K\sqrt{|x|}\log|x|.$$

Applying Claim 3,

$$\begin{aligned}\mathbf{E}G(2x) &\leq 2\mathbf{E}G(x)+C_1\sqrt{|x|}\log|x| \\ &< 2g(x)-(2K-C_1)\sqrt{|x|}\log|x| \\ &= g(2x)-(2K-C_1)\sqrt{|x|}\log|x|.\end{aligned}$$

If this is the case for $x = nx$ for infinitely many n, then along a subsequence we would find

$$\mathbf{E}G(2nx)/n \to \ell < g(2x);$$

and impossibility as per (1). Hence, the second inequality holds.

The lemma is proved. ☐

Consequently, we have

Corollary 4. *If $|x|$ is sufficiently large and $s > |x|^{1/2+\epsilon}$, then*

$$\mathbf{P}(|G(x)-g(x)|>s) < \exp(-cs^2/|x|).$$

Proof. By the triangle inequality and Lemma 2,

$$\begin{aligned}\mathbf{P}(|G(x)-g(x)|>s) &\leq \mathbf{P}(|G(x)-\mathbf{E}G(x)|>s-|\mathbf{E}G(x)-g(x)|) \\ &\leq \mathbf{P}(|G(x)-\mathbf{E}G(x)|>s-C_1\sqrt{|x|}\log|x|)\end{aligned}$$

Taking $|x|$ sufficiently large and assuming $s > |x|^{1/2+\epsilon}$, so that $|x|^\epsilon - C_1\sqrt{|x|} \approx |x|^\epsilon$ and $s > C_1\sqrt{|x|}\log|x|$; we obtain the result by applying (4). ☐

Finally, putting everything together, we can show that $\mathbf{P}(0 \to y \to x)$ has stretched exponential decay.

Suppose $y = (\alpha x_1, \beta x_2)$ satisfies

$$\sqrt{\alpha\beta}+\sqrt{(1-\alpha)(1-\beta)} < 1 - \frac{|x|^{\epsilon+2/3}}{2\sqrt{x_1x_2}}$$

(this corresponds to $(y_1, y_2) < x$ outside a small region around the line from 0 to x) so that

$$g(x)-(g(y)+g(x-y)) > |x|^{\epsilon+2/3} \gg |x|^{\epsilon+1/2}.$$

By Corollary 4, the probability that any one of $|G(x)-g(x)|$, $|G(y)-g(y)|$ or $|G(y,x)-g(x-y)|$ is larger than $|x|^{\epsilon+1/2}$ is less than $\exp(-c|x|^{2\epsilon})$. Hence recalling that $0\to y\to x$ occurs if and only if $G(x)=G(y)+G(x,y)$, we have

$$\mathbf{P}(0\to y\to x)\lesssim \exp(-c|x|^{\beta}) \tag{7}$$

for some $\beta>0$.

2.2 An infinite last passage path is asymptotically straight

Using the results of the previous subsection, we can show that any geodesic in the LPP tree has an asymptotic direction.

In what follows let L_α denote the straight line passing through the origin with slope $\alpha\in(0,\pi/2)$. We will show that a.s. only finitely many $x\in L_\alpha$ have some $y_x\in L_\gamma$ ($\gamma\neq\alpha$) such that $0\to x\to y_x$.

Suppose $|x|\le|y|\le 2|x|$ and $\theta=|\arg(y)-\arg(x)|\gtrsim|x|^{-\delta}$. (See Figure 4.) Then by (7), $\mathbf{P}(0\to x\to y)\lesssim\exp(-|x|^{\beta})$, for some $\beta>0$. Thus, for such y, w.h.p. the event $0\to x\to y$ does not occur. So taking the union bound, and applying Borel-Cantelli, we see that a.s. for all but finitely many y outside the θ-cone corresponding to x (see Figure 4) the event $0\to x\to y$ does not occur.

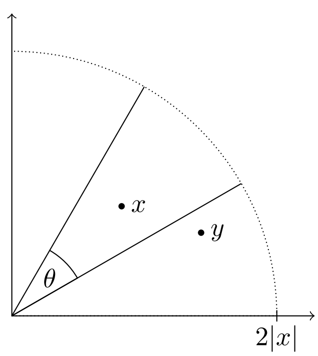

Figure 4: θ-cone corresponding to x.

Iterating the above argument, we conclude

Lemma 5. *For sufficiently large $|x|$, a.s. all z such that $0\to x\to z$ satisfy*

$$|\arg(z)-\arg(x)|\lesssim|x|^{-\delta}.$$

Proof. Using translation invariance, the above argument can be generalised: we see that a.s. for all but finitely many z with $|x|\le|z|\le 2^k|x|$, if $0\to x\to z$, then

$$|\arg(x)-\arg(z)|\lesssim\sum_{i=0}^{k-1}|2^i x|^{-\delta}\lesssim|x|^{-\delta}.$$

Since the above holds for all k, we obtain the result. $\square$

Finally, we see that infinite paths in the LPP tree are asymptotically straight.

Corollary 6. *Every geodesic in the LPP tree has an asymptotic direction.*

Proof. Suppose $\{x_n\}$ are connected by an infinite LPP path. Then, by Lemma 5, $\{x_n/|x_n|\}$ is a Cauchy sequence, and so converges to some limit. ☐

In fact, more is true:

Theorem 7. *The last passage percolation tree a.s. satisfies the following:*

1. *There exists an infinite geodesic in every direction in* $[0, \pi/2]$.
2. *There is a dense, countable set of directions with two infinite geodesics. (These correspond to infinite geodesics in the dual tree.)*
3. *There is no direction with three distinct infinite geodesics.*

Proof. (Property 1.) Consider geodesic paths from 0 to $\lfloor nx \rfloor$ for $n \in \mathbb{N}$. By arguments from this subsection we know that a.s. the geodesic visits only finitely many points outside a small wedge containing $\lfloor nx \rfloor$. Taking the pointwise topology on geodesics, we obtain a compact space. Hence, there is some infinite collection $\{n_k\} \subset \mathbb{N}$, for which the sequence of geodesics to $\lfloor n_k x \rfloor$ converges in the direction of x. (Note that this does not rule out the possibility of two subsequences with geodesics converging along L_x; one from above and one from below. This leads us to property 2.)

(Property 2.) By Property 1 there are infinite geodesics in the LPP tree converging in all directions. Between every two such geodesics there is at least one infinite geodesic in the dual tree. Hence there is a dense set of directions to which dual geodesics converge. Each such direction corresponds to an interface between two geodesics in the LPP tree, i.e. a direction to which two geodesics in the tree converge; one from above and one from below. This set of directions is countable since every infinite geodesic is the dual tree begins at a point in the plane with rational coordinates. ☐

Note: We do not prove Property 3 in Theorem 7 as its proof requires ideas not presented in these lectures.

2.3 Generalising to other domains

In the next lecture we will consider LPP paths in other domains than $\mathbb{Z}_+^2$.

Let $S_0 \subset \mathbb{Z}^2$, and let $G(S_0, x)$ be the maximum weight of a path from S_0 to $x \notin S_0$. It turns out that in this more general context, the union of paths is a forest, i.e. a union of trees. To analyse this setting we will come back to TASEP dynamics and second class particles.