Lecture of 2012-06-04 **scribe: Gourab Ray**

We shall give a proof of Burke's theorem. Let $A(s,t)(D(s,t))$ denote the number of arrivals (departures) between times s and t.

Theorem 1 (Burke)**.** *Let A and S denote the arrival process and the service process of an $M/M/1$ queue at equilibrium with rates λ and μ respectively with $\lambda < \mu$. Then the departure process D is poisson with rate λ.*

Proof. The process $Q(t) = \sup_{s\le t}[A(s,t) - D(s,t)]^+$ is a reversible process. Recall that a CTMC with state space $\mathbb{Z}$ is time reversible if we have for all $(i,j) \in \mathbb{Z}^2$ we have $p_i q_{ij} = p_j q_{ji}$ where $\{q_{ij}\}$ are the transition rates and $\{p_j\}$ is the stationary distribution. Note that $q_{ii+1} = \lambda$, $q_{ii-1} = \mu$ and $q_{ij} = 0$ otherwise. Now it is easy to see that $p_i = c_{\mu\lambda}(\lambda/\mu)^{i-1}(\lambda+\mu)^{-1}$ for $i \ge 1$ and $p_0 = c_{\mu\lambda}\mu/(\lambda+\mu)$ for some proper normalizing constant $c_{\mu\lambda}$. From this one can check that the condition for reversibility is satisfied. So the number of decreases of Q which are precisely the departures are equal in distribution to the number of arrivals which are the number of arrivals of Q (just look at the process with time running backwards). This completes the proof as the arrival process is a poisson process with rate λ. $\square$

Remark 2. The proof also shows that since the arrival process on $[t,\infty)$ is independent of $Q(t)$ (by memorylessness), the departure process on $(-\infty,t]$ is independent of $Q(t)$.

0.1 Shape of LPP

We proved that $G(x+x_0) - G(x_0)$ converges to some limit as x_0 goes to infinity in some direction. Recall that if

$$\begin{aligned} X(u) &\sim \exp(\alpha) \text{ if } u \in \{0\}\times\mathbb{Z}^+ \\ &\sim \exp(1-\alpha) \text{ if } u \in \mathbb{Z}^+\times\{0\} \\ &\sim \exp(1) \text{ if } u \in (\mathbb{Z}^2)^+ \end{aligned}$$

Let $G_0(x) = \max_{\gamma:0\to x}\sum_{w\in\gamma} X(w)$. We have already seen

$$G_0(x) = \frac{x_1}{1-\alpha} + \frac{x_2}{\alpha} + o(||x||) \tag{1}$$

Now note that

$$G_0(x) = (\max_{k\le x_1}(G_0(k,0) + (G((k,1),x)) + X(k,1)) \vee (\max_{k\le x_2}(G_0(0,k) + (G((1,k),x)) + X(1,k))$$

the above equation is just stating the fact that the longest path to x must leave the X or Y axis at some point and after that the path takes up the same weights on its way as an LPP. Fixing $x = (n,n)$ we get from (1) that

$$\frac{n}{1-\alpha} + \frac{n}{\alpha} + o(n) = \max_{k\le n}\left(\frac{k}{1-\alpha} + G((k,1),x)\right) \vee \max_{k\le n}\left(\frac{k}{\alpha} + G((1,k),x)\right) + o(n)$$

Note that the maximum of $X(k,n)$ is of logarithmic order as the maximum is taken over n variables. Now dividing by n and taking the limit, we get

$$\frac{1}{1-\alpha} + \frac{1}{\alpha} = \sup_{s\le 1}\left(g(1,1-s) + \frac{s}{1-\alpha} \vee \frac{s}{\alpha}\right)$$

This follows from the convergence theorem **??** and the fact that g is symmetric about the line $y = x$. So we get the following equation for $\alpha \in [0, 1/2]$:

$$\frac{1}{1-\alpha} + \frac{1}{\alpha} = \sup_{s\le 1}\left(g(1,1-s) + \frac{s}{\alpha}\right) \tag{2}$$

Now to solve the equation (2), we need the theory of convex duality. For a convex function $f : \mathbb{R} \to \mathbb{R} \cup \{\infty\}$, we define the convex dual f^* as follows:

$$f^*(y) = \sup_x \{xy - f(x)\}$$

If f is continuous, it can be proved that $f^{**} = f$ (exercise).

Let $t = 1 - s$. Then $f(t) = -g(1,t)$ is convex. From (2), we get

$$\frac{1}{1-\alpha} + \frac{1}{\alpha} = \sup_{t\le 1}\left(-f(t) + \frac{1-t}{\alpha}\right) = f^*(-\frac{1}{\alpha})$$

which simplifies to

$$f^*(-1/\alpha) = 1/(1-\alpha).$$

So $f^*(y) = y/(y+1)$. From convex duality,

$$f(x) = \sup_y (xy - y/(y+1)) \tag{3}$$

The function $xy - y(y+1)$ maximises at $y = -1 + 1/\sqrt{x}$. Plugging in, we have

$$f(x) = (1 + \sqrt{1-t})^2$$

where $t = -1/x$.

0.2 TASEP with general initial conditions

For any general intial height function, the slope of the function at any given time at any position approximately determines the density of particles present in that position at that time. Namely if the slope is $q \in [-1, 1]$, then the density is given approximately by $p = (1-q)/2$. Now locally we might assume that the process has achieved stationarity. Hence for i.i.d steps we know that the rate of increase is roughly given by $2p(1-p) = (1-q^2)/2$. So, we believe

$$\frac{\partial}{\partial t} h_t(x) = \frac{1 - (h_t')^2}{2} \tag{4}$$

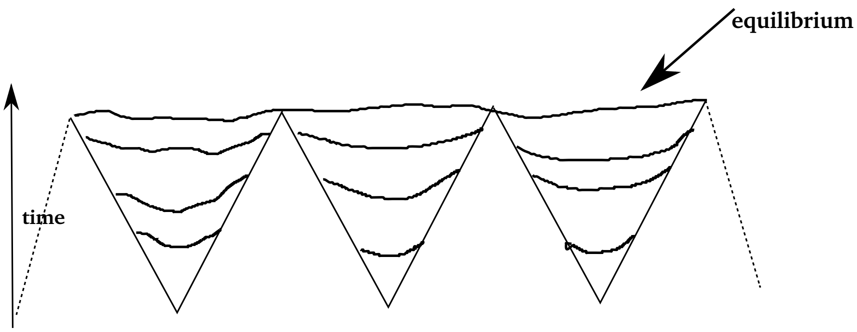

Figure 1: Growth process in a self similar structure. Locally the process grows like a parabole given by solutions to the initial function being $|x|$ ultimately converging to an equilibrium.

Equation (4) is known as the Burger's equation. By similar heuristics, if $U_t(x)$ denote the density of particles in position x at time t, Burger's equation takes the form:

$$\frac{\partial}{\partial t}U_t(x) = -\frac{\partial}{\partial x}(U(1-U))$$

Our goal is to obtain a scaling limit of the form

$$\frac{1}{k}h_{kt}(kx) \to \text{limit shape}$$

for some general initial function $h_0(x)$. This type of limit is known as the *hydrodynamic limit* in the literature. Here is a theorem which sheds some light on the problem.

Theorem 3. *Suppose $h_{kt}^{(k)}(kx)/k$ converges as k goes to infinity to some function $f_0(x)$, then*

$$\frac{1}{k}h_{kt}^{(k)}(kx) \to f_t(x)$$

as $k \to \infty$ where f satisfies burger's equation (4).

Remark 4. For corner growth with $h^0(x) = |x|$ is self similar, that is, $h_{kt}^{(k)}(kx)/k = |x|$. We can also consider functions not self similar which will grow like the Figure 1

Remark 5. The scaling here is also kind of special. For example if we consider simple random walk without exclusion on the line, then $U_t(x)$ is roughly given by $U_0 * p_t(x)$ where $p_t(x)$ is the gaussian kernel: $1/\sqrt{2\pi t}\exp(-x^2/2t)$. So if $1/kU_0^{(k)}(kx)$ converge to $f_0(x)$, then $1/kU_{k^2t}^{(k)}(kx)$ converge to $f_t(x)$ where $f_t(x) = f_0(x) * p_t(x)$.

Suppose $\{h_0^i(x)\}_{i\in I}$ be a set of height functions for some index set I. Let $g_0(x) = \inf_i h_0^i(x)$. Define $g_t(x)$ and $\{h_t^i(x)\}$ using the standard coupling.

Lemma 6. *If* $\{g_t(x)\}_{t\geq 0}$ *and* $\{h_t^i(x)\}_{i\in I, t\geq 0}$ *be defined as above. Then*

$$g_t(x) = \inf_i h_t^i(x)$$

Assuming the lemma we can now derive a form of the limiting shape for a general initial condition. Given g_0 define $h_0^i := g_0(i) + |x - i|$. So it is clear that $g_0 = \inf_i h_0^i$. But we know that $h_t^i(x) = g_0(i) + \phi_t(x - i) + o(t)$ where ϕ_t is the solution for the TASEP with the initial condition $h_0(x) = |x|$ as proved earlier. Hence from Lemma 6 we get that

$$g_t(x) = \inf_i \{g_0(i) + \phi_t(x - i) + o(t)\}$$

This is known as the variational solution of the Burger's equation.

Proof of Lemma 6. Since $g_0 \leq h_0^i$, we have $g_t \leq h_t^i$ from the monotonicity of the standard coupling. Suppose we make a step at x in time t and the hypothesis is satisfied upto time t. Then if $g(x)$ has increased, we are fine because of monotonicity. Otherwise either $g_t(x+1) = g_t(x) - 1$ or $g_t(x-1) = g_t(x) - 1$ or both. Then for some i So $h_t^i(x+1) = h_t^i(x) - 1$ or $h_t^i(x-1) = h_t^i(x) - 1$ or both. Hence h_t^i is also not increased. Hence the proof. □

0.3 Density of particles

Now let us give a brief summary of what happens when we look at the density of particles U_t at time t with different initial configurations of densities U_0 (known as steps). The initial corner growth process corresponded to a density function

$$\begin{aligned} U_0(x) &= 1 \text{ if } x < 0 \\ &= 0 \text{ if } x > 0 \end{aligned}$$

In this case we know that the limit shape is a parabola. So the density at time t is given by

$$\begin{aligned} U_t(x) &= 1 \text{ if } x \geq t \\ &= 0 \text{ if } x \leq -t \\ &= \frac{t - x}{2t} \text{ if } -t < x < t \end{aligned}$$

But there is a big difference if we start with a decreasing step or an increasing step. Suppose we start with the density $f_0(x) = \alpha$ if $x < 0$ and $f_0(x) = \beta$ if $x \geq 0$ where $\beta < \alpha$. Then we get a linear spread with slope $1/2t$ (according to the Burger's equation). Thus we get

$$\begin{aligned} U_t(x) &= \beta \text{ if } x \geq at \\ &= \alpha \text{ if } x \leq -bt \\ &= \frac{t - x}{2t} \text{ if } -bt < x < at \end{aligned}$$

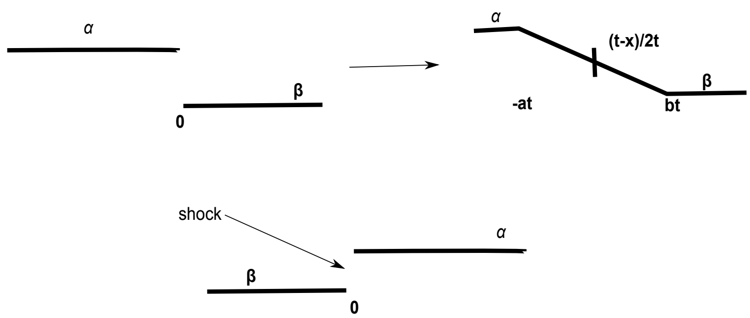

Figure 2: Density profiles of TASEP with different initial conditions

where $(1-a)/2 = \alpha$ and $(1-b)/2 = \beta$. If on the other hand we have an increasing step, that is, $\beta > \alpha$ then the discontinuity is known as a "shock" (see figure 2). We know that if the density was identically β or α, it would have been stationary. However now the "shock" will move at a constant speed of $1+\alpha-\beta$ to the right.