Lecture of 2012-06-05 **scribe: Alex Tomberg**

1 Last passage percolation

Recall the last passage percolaiton model: Let X_u be i.i.d. random variables for $u \in \mathbb{Z}^d$. We assume throughout that a.s. $X_u \geq 0$. We write $u \leq v$ for $u, v \in \mathbb{Z}^d$ if $u_i \leq v_i$ for all $i = 1, 2, \ldots, d$. Define the last passage time for $u \leq v$

$$G(u, v) = \max_{\gamma : u \to v} \sum_{w \in \gamma} X_w,$$

where the maximum is over all monotone increasing paths from u to v. For certain computations it is convenient to assume that the sum over γ includes its end point v but not its starting point u. We shall abbreviate $G(x) = G(0, x)$ in some cases.

We shall see that G has a deterministic asymptotic value, in the following sense.

Theorem 1. *There exists a function $g : \mathbb{R}_+^d \to \mathbb{R} \cup \{\infty\}$, depending only on the law of X_u, such that a.s. for every $x \in \mathbb{R}_+^d$,*

$$\frac{1}{n} G(nx) \xrightarrow[n \to \infty]{} g(x).$$

Either $g = \infty$ for all $x \in (0, \infty)^d$, or else it is finite everywhere. Moreover, g is continuous, increasing in each coordinate, homogeneous ($g(ax) = ag(x)$ for $a > 0$), superadditive ($g(x + y) \geq g(x) + g(y)$) and concave.

Recall that G is monotone, which implies monotonicity of g. Homogeneity follows from the form of the limit, given that it exists. Together with monotonicity, homogeneity implies continuity in $(0, \infty)^d$: the box with corners $(1 - \delta)x, (1 + \delta)x$ is a neighbourhood of x in which the values of g are within factor $\delta g(x)$ of $g(x)$.

To see that g is superadditive, namely $g(x + y) \geq g(x) + g(y)$, note that

$$\geq \frac{1}{n} G(0, nx + ny) \frac{1}{n} G(0, nx) + \frac{1}{n} G(nx, nx + ny).$$

A.s. the LHS converges to $g(x+y)$ and the first term on the RHS to $g(x)$. While we have not proven a.s. conergence of the second term to $g(y)$, we have from translation invariance of last passage percolation

$$G(nx, nx + ny) G(0, ny).$$

Therefore $\frac{1}{n} G(nx, nx + ny)$ converges to $g(y)$ in distribution. It follows that it converes a.s. along a subsequence, yielding the desired inequality.

Figure 1: Splitting of the box of size n into smaller boxes of size $p^{-1/d}$.

Concavity is just a combination of superadditivity and homogeneity:

$$g(\alpha x + \bar{\alpha} y) \geq g(\alpha x) + g(\bar{\alpha} y) = \alpha g(x) + \bar{\alpha} g(y).$$

Finally, since g is monotone, if $g(x) = \infty$ for some $x \in (0\,infty)^d$, then $g(y) = \infty$ for all $y \geq x$, and by homogeneity, on all of $(0, \infty)^d$.

The case $g \equiv \infty$ is indeed possible, even if $\mathbb{E}X_u < \infty$ (so that each particular path still has total sum of order $\|x\|$. It is a generally open problem to find a necessary and sufficient condition on the law of each X_u so that $g < \infty$. One sufficient condition for $g < \infty$ is given by following theorem of [**?**]

Theorem 1. *In $\mathbb{Z}^d$, if for some $\epsilon > 0$,*

$$\mathbb{E}\left[X^d \log^{d+\epsilon} X\right] < \infty,$$

then $g < \infty$.

This result is tight in that if X doesn't have finite dth moment, then $g \equiv \infty$:

Proposition 2. *If $\mathbb{E}\left[X^d\right] = \infty$, then $g \equiv \infty$.*

Proof. First note that

$$G(n,n) \geq \max_{u \leq (n,n)} X_u$$

Now, if $\mathbb{E}X^d = \infty$, then for any constant c and infinitely many n, there is $u \leq (n,n)$, suth that $X_u \geq cn$. Therefore G cannot be constant, so it cannot be concentrated either. □

Theorem 2. *If*

$$\int_0^\infty \mathbb{P}(X > t)^{1/d} dt < \infty,$$

then $g < \infty$.

Lemma 3. *Let $X_u = \text{Bernoulli}(p)$, then*

$$\mathbb{E}\left[G(n,n)\right] \lesssim np^{1/d},$$

where $\lesssim$ means and inequality up to a constant that may depend on d, but not on p.

Proof. Note that when p is large the statement in easy, so we will only consider the case when $p \to 0$. Then, after rescaling, we obtain a Poisson process.

We split the box into smaller boxes of size $p^{-1/d}$ as in figure 1. The number of 1's in each box coverges to Poisson(1) as $p \to 0$.

$$\mathbb{E}\left[G(n,n)\right] \leq \mathbb{E}\left[G_{\text{poisson}}(np^{1/d})\right] \leq np^{1/d} g_{\text{poisson}}((1,1)) < \infty.$$

The finiteness of g_{poisson} follows from the fact that the Poisson distribution has a very fast decaying tail. □

Proof of Theorem. Since

$$X = \int_0^\infty \mathbb{1}_{X>t} dt,$$

we have

$$G(0,x) = \max_{\gamma:0\to x} \sum_\gamma \int_0^\infty \mathbb{1}_{X_u>t} dt.$$

So, by Fubini,

$$\begin{aligned}\mathbb{E}\left[G(0,x)\right] &= \mathbb{E}\left[\max_{\gamma:0\to x} \sum_\gamma \int_0^\infty \mathbb{1}_{X_u>t} dt\right] \\ &\leq \int_0^\infty \mathbb{E}\left[\max_{\gamma:0\to x} \sum_\gamma \mathbb{1}_{X_u>t}\right] dt.\end{aligned}$$

Now, $\mathbb{1}_{X_u>t}$ is a Bernoulli random variable, so we obtain the estimate

$$\mathbb{E}\left[G(0,x)\right] \leq \int_0^\infty \mathbb{E}\left[G_{\mathrm{bernoulli}(p_t)}(0,x)\right] dt$$

where $p_t := \mathbb{P}\left[X_u > t\right]$.

Finally, if we set $x = n\mathbb{1}$ and use the result of Lemma 3, we get

$$\frac{1}{n}\mathbb{E}\left[G(0,x)\right] \lesssim \int_0^\infty \mathbb{P}(X > t)^{1/d} dt. \qquad \square$$

The problem of finding a necessary and sufficient condition for $g < \infty$ remains open.

Remark 4. Let $\tilde{X} := X \wedge M$, for some constant M. The above argument gives us that

$$0 \leq g(\mathbb{1}) - \tilde{g}(\mathbb{1}) \lesssim \int_M^\infty \mathbb{P}(X > t)^{1/d} dt.$$

The above remark allows us to approximate general weights by truncated ones.

For example, the following theorem could be used to say something more about $\tilde{g}(\mathbb{1})$ than what we already know about $g(\mathbb{1})$.

Theorem 3 (Talagrand, Martin)**.** *Given sets C_i of size less than R, we consider iid weights Z_u bounded by M, i.e. $|Z_u| < M$. If*

$$Q = \max_i \sum_{u\in C_i} Z_u,$$

Then

$$\mathbb{P}\left[|Q - \mathbb{E}Q| > s\right] \lesssim \exp\left[-\frac{cs^2}{M^2R}\right].$$

A typical application of this theorem is to conclude that $G(nx)$ is within $\sqrt{n}$ of $\mathbb{E}[G(nx)]$, which is about $ng(x)$.

Conjecture For a bounded X_u,

$$\frac{G(nx) - ng(x)}{C_x n^{1/3}} \stackrel{n\to\infty}{\longrightarrow} F_2,$$

where F_2 is a Tracy-Widom distribution (maximum eigenvalue of a Gaussian Unitary Matrix) and C_x is a constant depending on x and the distribution of X_u.

Fluctuations of $G(nx)$ are thus of order $n^{1/3}$ and so, it is easy to see that the fluctuations of the limiting shape are also of the order $n^{1/3}$. To be more precise, the fluctuations of $h_t(x)$ in t are of the order $n^{1/3}$, while the correlations in x are of the order $n^{2/3}$. Therefore,

$$\left\{n^{-1/3}h(n^{2/3}t)\right\}_t \stackrel{(weakly)}{\Longrightarrow} A(t).$$

This is a general property of the KPZ univerality class, where our corner growth model lies.

- Is g continuous at the boundary?
- If $\int_0^\infty \mathbb{P}(X > t)^{1/d}dt < \infty$, then g the answer to the previous question is yes, but is the weaker condition $g < \infty$ sufficient?
- Is g strictly concave? Generally no. In fact for the Bernoulli(p) case, there is a range of x such that $g(x) = \|x\|_1$, so that the level sets of g are *not* strictly convex. This also implies that g itself is not strictly concave.

Theorem 4. *Let $S_t = \{x : G(0, x) < t\}$. If $\int_0^\infty \mathbb{P}(X > t)^{1/d}dt < \infty$, then*

$$\frac{1}{t}S_t \longrightarrow \{x : g(x) < 1\}.$$

Proof. We prove the statement by using concentration inequalities and Borel-Cantelli lemma.

By concentration (for truncated version), fo a fixed $\epsilon > 0$,

$$\mathbb{P}\left[|G(x) - \mathbb{E}G(x)| > \epsilon \|x\|\right] \lesssim \exp\left(-\frac{c\|x\|^2\epsilon^2}{M^2\|x\|}\right).$$

Since the RHS is summable over the whole quadrant, the even only occurs finitely many times for all $\epsilon > 0$ by Borel-Cantelli. □

So, the limit shape is known for $X_u \stackrel{(d)}{=} \text{Exp}(1)$, and in fact for

$$X_u \stackrel{(d)}{=} \text{Geom}(p), \text{ where } \mathbb{P}(X = k) = p\bar{p}^{k-1}.$$

In the exponential case it is $g(x, y) = (\sqrt{x} + \sqrt{y})^2$, and for the geometric case,

$$g(x, y) = \frac{1}{p}(\bar{p}(x + y) + 2\sqrt{xy\bar{p}}).$$

These results come from the memoryless property of the exponential and geometric distribution, which makes S_t into a Markov Chain. Studying Markov Chains requires the knowledge of their stationary distributions. We will see that for every $\rho \in [0, 1]$, there is a stationary measure μ_ρ, which is a product measure. Under μ_ρ, every site is occupied with probability ρ independently of the others.