Lecture of 2012-06-04 **scribe: Gourab Ray**

1 The TASEP and several equivalent processes

We are going to consider a system of particles which interact with each other. The setting might be very general where the particles will do random walk on a graph and interacting in a nontrivial manner. For the moment we are concerned with a *simple exclusion process* which is a process in which jumps to occupied sites are suppressed. To simplify things we take the graph to be $\mathbb{Z}$. We consider continuous version of this process in which particles jump to left or right at some rate. Let the particle jump to the right with rate p and to the left with rate $1-p$. If $p \neq 1/2$ the process is called *asymmetric simple exclusion process* (ASEP). If $p = 1/2$, its called a *symmetric exclusion process.* If $p = 1$ the process is called a *totally asymmetric simple exclusion process* or TASEP in short. We are mainly going to deal with TASEP for a while. However we need to show first that it exists. It will be helpful to consider the graphical construction of TASEP for this.

Notation: $\overline{p} = 1 - p$.

1.1 Graphical Construction (Harris)

We shall construct the EP for general graphs with $p_c > 0$. We consider $G \times \mathbb{R}^+$. Let $E(G)$ be the set of oriented edges of G and $V(G)$ be the set of vertices. For every oriented edge $e \in E$ we associate an independent family of Poisson processes $\{P_e\}_{e \in E}$ with intensity λ_e. We assume $\sup_{e \in E} \lambda_e = \lambda < \infty$ For each e for every jump time τ for P_e, we add the oriented edge e in $V(G) \times \{\tau\}$. For any initial configuration of particles on $\mathbb{Z}$, to know the position of the particle at any time t with the initial position being v, we shall start from v and look at the first time in there is an oriented edge e incident to $\{v\} \times \mathbb{R}^+$ going away from v. Then we try to move the particle along e to the neighbour w of v if the position w is unoccupied. Certainly in a finite graph this construction makes sense, However in an infinite graph there may be simultaneous jumps so we might run into problems.

However we might get past this if we apply the following trick: let t_0 be small enough such that $e^{-\lambda t_0} > 1 - p_c$ where p_c is the critical edge percolation probability for G. Then of course, the edges where there is any activity upto time t_0 form finite clusters almost surely. We can define the process upto time t for the finite clusters separately. Then again apply the fact the Poisson process is memoryless to complete the construction for any $t > 0$. In particular for ASEP on $G = \mathbb{Z}$ we just take $\lambda_{(x,x+1)} = p$ and $\lambda_{(x,x-1)} = \overline{p}$.

N.B. Things could however break down for $p_c = 0$ (consider a tree whose degree at level i is $i + 1$ for example for $i \geq 0$). We are not going to consider such situtations.

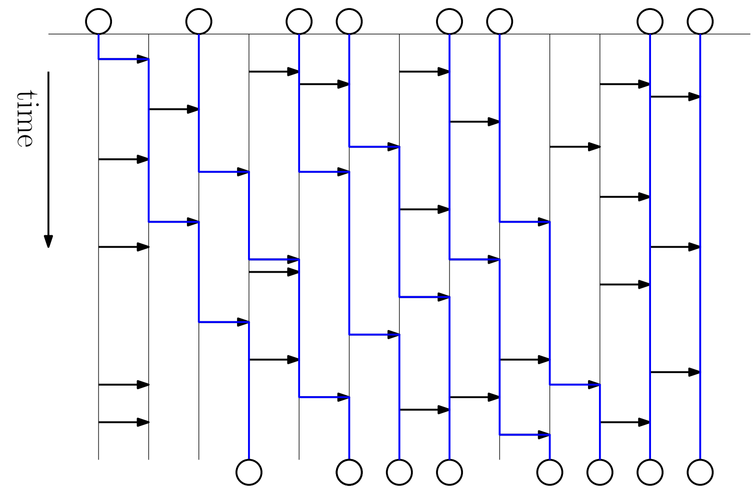

Figure 1: Graphical construction of TASEP on a finite line. The circles are particles and dots are holes.

1.2 TASEP as growth process

Let $\xi_t \in \{0,1\}^{\mathbb{Z}}$ be the state at time t, where we interpret 1 as a particle 0 as a "hole"(disclaimer: sometimes we might consider $\{1,\infty\}^{\mathbb{Z}}$ or in more general scenario $S^{\mathbb{Z}}$ for some finite alphabet set S). The process proceeds as follows: if $\xi_t(i) = 1$ and $\xi_t(i+1) = 0$ then we swap at rate 1, that is after an $\exp(1)$ amount of time τ, $\xi_{t+\tau}(i+1) = 1$ and $\xi_{t+\tau}(i) = 0$. From the conguration ξ_t, we can consider the height function $h_t : \mathbb{R} \to \mathbb{Z}$ for $t \geq 0$ which we shall define now. Let $h_0(0) = 0$ and for $i \in \mathbb{Z}$, $h_0(i+1) - h_0(i) = -1$ if $\xi_0(i) = 1$.

So if we have a particle followed by a hole, that is, $h_t(i+1) = h_t(i-1) = h_t(i) + 1$, then $h_t(i)$ changes to $h_t(i) + 2$ at rate 1. So pict orially we see

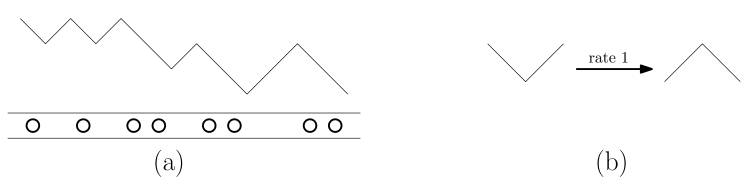

Figure 2: (a)The height function h_t. circles denote particles and dots denote holes. Dots correspond to an upstep in the height function and circles corrsspond to downstep. (b) The flipping of corners happening at rate 1

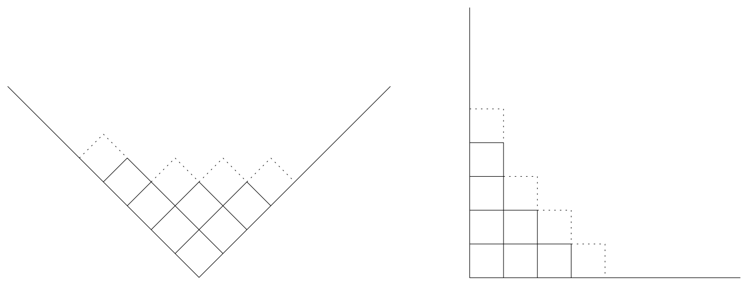

Figure 3: (a)The corner growth process in russian co-ordinates. (b) We consider the growth process obtained by flipping the figure by 45°

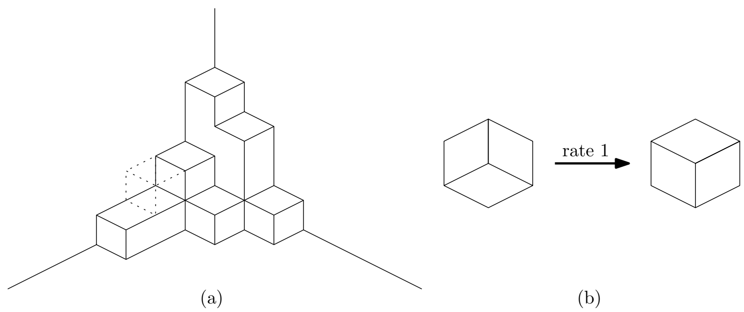

Figure 4: (a) The 3-d corner growth process for 2-dim TASEP (b) The lozenge tiling reprsentation

a corner pointing down changes to corner pointing up with rate 1 (see figure 2(b)). We will be considering TASEP with the special initial configuration where all the particles are occupying sites indexed by non-positive integers. The corresponding height function is $h_0(x) = |x|, x \in \mathbb{R}$. The growth process can be thought of as adding boxes on the graph of $h_0(x)$ randomly. The boxes are added at sites where it is possible to add at rate 1.

More formally, let us define the growth process by considering the picture obtained by rotating the figure by 45°. More formally, let e_1, e_2 be the canonical basis vectors for $\mathbb{R}$. If ($x - e_1$ and $x - e_2$) are both present then we add the square with vertices $(x, x-e_1, x-e_2, x-e_1-e_2)$ at rate 1. In particular the rate at which the vertices are added depend upon the number of positions where we can add a square (or positions on $\mathbb{Z}$ where particles can move).

We can also consider a 2-dim version of the TASEP where only up and right movements are allowed if possible. Then if we start with particles everywhere on $\mathbb{Z}^2$ except the first quadrant, we gat a 3-d version of the height function which when rotated gives us a corner growth version of the model in 3-d (see Figure 4(a)). The model is nothing but adding 3 dimensional boxes in the corners wherever possible at rate 1. Yet another way to look at this is to consider the projection of this in 2-d plane. Then we are going to have tilings of lozenges where the dynamics change as shown in Figure 4(b).

1.3 TASEP as Last Passage percolation (LPP)

Let $G : (\mathbb{Z}^2)^+ \to \mathbb{R}$ be a function such that $G(u)$ is the time to add u in the corner growth process. Define $G(n,0) = G(0,n) = 0$ for $n \in \mathbb{N}$. For $u \in (\mathbb{Z}^2)^+$ let X_u be the time to add u after $(u-e_1, u-e_2)$ are present. So by definition of TASEP, $X_u \stackrel{d}{=} \exp(1)$. So we have the recursion

$$G(u) = G(u-e_1) \vee G(u-e_2) + X_u \tag{1}$$

This is exactly a special case of the last passage percolation model or LPP in short. Here we have iid variables X_u for each $u \in (\mathbb{Z}^d)^+$. For a path γ, we consider $l(\gamma) = \sum_{u\in\gamma} X_u$. Let $\overrightarrow{p(u,v)}$ be the set of oriented paths from u to v (by oriented paths we mean paths going either right or up). for any u we consider the longest path from 0 to u and we let $G(u)$ (by abuse of notation) be its length. That is,

$$G(u) = \max_{\gamma \in \overrightarrow{p(0,u)}} \sum_{v\in\gamma} X_v$$

It is clear that $G(u)$ satisfies (1). So for X_u being $\exp(1)$, the TASEP is a special case for LPP (note here that we can consider maximal distance from origin without loss of generality (why?)).

Let $S_t = \{u \in \mathbb{Z}^d, G(u) < t\}$. we are concerned with a the limiting behaviour of this shape. This is given by the following theorem:

Theorem 1.

$$S_t/t \to B$$

for some set B almost surely, where the distance between the sets is in hausdorff metric(?)

The shape B depends upon the initial configuration. In "russian" co-ordinates, if $h_0(u) = |u|$, then $h_t(u)$ is give by a parabola with axes at the tangent. More precisely,

$$\begin{aligned} h_t(u) &\approx |u|; \quad \text{if } |u| > t \\ &\approx au^2 + b \quad \text{if } |u| \le t \end{aligned}$$

where a and b are given by $at^2 + b = t$ and $2at = 1$ (check!). In normal co-ordinates the boundary of the limit shape is given by the curve $\sqrt{x} + \sqrt{y} = \sqrt{t}$ (so the limit shape is the "$L^{1/2}$ ball"). From the shape we can also say something about the location of the kth particle at time t which is given by something known plus a $o(t)$ error. The following theorem settles the issue of existence of limit.

Theorem 2. *There exists a function $g : (\mathbb{R}^2)^+ \to \mathbb{R}^+ \cup \{\infty\}$ such that*

$$\frac{1}{n} G(\lfloor nx \rfloor) \to g(x)$$

almost surely simultaneously for all $x \in (\mathbb{R}^2)^+$

Proof. The proof follows from subadditive ergodic theorem and monotonicity of the function G. Suppose, we have a set of doubly indexed random variables $\{X(m,n)\}_{m\in\mathbb{N},n\in\mathbb{N}}$ such that,

$$X(m,n) + X(n,s) \geq X(m,s)$$
$$X(m,n) \stackrel{d}{=} X(m+k,n+k) \text{ (translation invariance)}$$

for nonnegative integers k. Then the subadditive ergodic theorem says that

$$\frac{X(0,n)}{n} \to Y$$

almost surely where Y is a constant. For $x \in \mathbb{N}^2$ we can define $Y(m,n) := G(mx,nx)$. Then it is easy to see that G is superadditive and translation invariant. We can still apply the subadditive ergodic theorem (why?) to see that simultaneously for all $x \in \mathbb{N}^2$, $1/nG(0,nx)$ converges to $g(x)$ for some function g. We can also do the same thing for the co-ordinates of x being rational (check!). We clearly see that $G(0,x) \leq G(0,y)$ where the co-ordinates of x re strictly less than y for any points x,y in the quarter plane. This fact allows us to prove the theorem simultaneously for all x by approximating the convergence on irrational points by that on rational coordinates and using the monotonicity (exercise). □