

Haruspicy and anisotropic generating functions

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Abstract

Guttman and Enting [Phys. Rev. Lett. 76 (1996) 344–347] proposed the examination of anisotropic generating functions as a test of the solvability of models of bond animals. In this article we describe a technique for examining some properties of anisotropic generating functions. For a wide range of solved and unsolved families of bond animals, we show that the coefficients of y^n is rational, the degree of its numerator is at most that of its denominator, and the denominator is a product of cyclotomic polynomials. Further, we are able to find a multiplicative upper bound for these denominators which, by comparison with numerical studies [Jensen, personal communication; Jensen and Guttman, personal communication], appears to be very tight. These facts can be used to greatly reduce the amount of computation required in generating series expansions. They also have strong and negative implications for the solvability of these problems.

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1. Introduction

1.1. Lattice animals

The enumeration of lattice animals is arguably one of the most famous problems in combinatorics and is of considerable importance in the study of lattice models in statistical physics and theoretical chemistry. Considering the intensive study that these models have been subjected to over their 40+ year history, it is perhaps a little surprising that the number of rigorous results is very small, and that the number of models that have been solved exactly, either implicitly or explicitly, is yet smaller.

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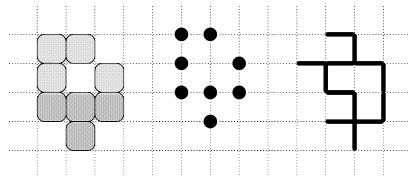


Fig. 1. Three basic types of lattice objects (from left to right): polyominoes, site animals and bond animals. Polyominoes and site animals are equivalent under lattice duality; the polyomino and site animal depicted here are equivalent to one another.

There are three basic types of lattice objects; polyominoes, site animals and bond animals. A polyomino is a finite connected set of lattice faces (or *cells*), a site animal is a connected set of lattice vertices (or *sites*) and a bond animal is a connected set of lattice edges (or *bonds*); all are defined up to translation (if two animals are equal under some translation, then we consider them to be the same animal). By replacing each cell of a polyomino with a site at its centre, one obtains an equivalent site animal on the dual lattice and *vice versa* (see Fig. 1). In this paper we will only study bond animals on the square lattice. We will write “*animal*” to mean “*square lattice bond animal*.”

The fundamental question in the study of animals is “*How many animals are there that contain n bonds?*” Let us denote the answer to this question by c_n . It is not difficult to compute c_n for small n , either with a pencil and paper or with a computer and a compiler, simply by listing every animal with n bonds. In principle one can do this for any n , but it quickly becomes obvious that c_n is growing rapidly with n and so the time it takes to compute it is also growing rapidly with n . It can be shown using a concatenation argument [15] that the limit

$$\lim_{n \rightarrow \infty} (c_n)^{1/n} = \mu$$

exists, and so the number of animals grows exponentially with n (i.e., $c_n \sim A\mu^n$, with sub-exponential corrections). So although brute force methods¹ always work, they are not very satisfactory, and so are not normally considered to be a solution.

More mathematically appealing are solutions in the form of (non-trivial) expressions for c_n , a recurrence for c_n that can be computed quickly, or (as we will concentrate on here) an expression for the generating function, $\sum c_n x^n$. Let us consider the enumeration of two families of bond animals: self-avoiding polygons and staircase polygons.

Example 1 (Staircase polygons). An animal is a *staircase polygon* if each row and column of the animal contains exactly 2 bonds, and further the lower and upper edges are directed paths (taking only north and east steps) that intersect exactly twice.

This model is well understood and we are able to compute c_n in a several different (equivalent) ways:

¹ For example, methods of the form: (1) input n , (2) list every animal with n bonds, (3) output the length of the list.

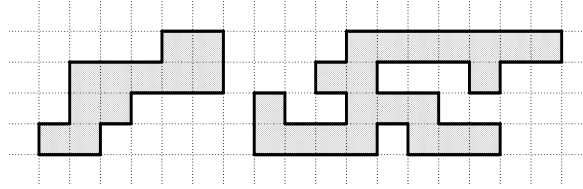


Fig. 2. A staircase polygon (left) and a self-avoiding polygon (right).

- the coefficient: $c_n = \frac{1}{n+1} \binom{2n}{n}$,
- the generating function: $\sum_{n \geq 1} c_n x^n = (1 - 2x - \sqrt{1 - 4x}) / (2x)$,
- c_n satisfies $(n + 1)c_n - (4n - 2)c_{n-1} = 0$, with $c_0 = 0$ and $c_1 = 1$.

Exact solutions (such as above) are only known for those models with severe topological restrictions: spiral walks (see [1], for example), three choice polygons [6], a number of families of column-convex polygons (see [2], for example), and three-dimensional convex polygons [3]. For other problems we have to use brute force, or other algorithms that are still exponential in time.

Example 2 (Self-avoiding polygons). An animal is a *self-avoiding polygon* (SAP) if it is the embedding of a simple closed loop into the square lattice (the vertices of the underlying graph are all of degree 2). Despite a great deal of effort over many years, no non-trivial expression for the number of SAPs with $2n$ bonds is known. Nor do we have a non-trivial expression for the corresponding generating function, nor do we know a non-trivial recurrence satisfied by these numbers.

There are a number of other interesting animal models (general bond animals, bond trees, directed bond animals, and self-avoiding walks, ...) that have proved to be equally difficult to count. To date, the best way of computing c_n for these models is the *finite-lattice method* which is an algorithm that requires exponential time and space, but is still exponentially faster than brute-force methods (see, for example, [7–9]).

The history of lattice animal enumeration suggests that attempts to find c_n for general families of animals are very likely to be frustrated. Rather than embarking on a (probably doomed) effort to find a solution, we seek to examine some of the properties of the solution. To do this we examine *anisotropic* generating functions; in particular, one observes a marked difference in the “structure” of the coefficients of anisotropic generating functions for solved and unsolved models [10,11]. The main task of this paper is to explain this structure and explore some of its consequences.

1.2. Anisotropic generating functions

The *isotropic* generating function of a family of animals enumerates the animals according to the total number of bonds. The *anisotropic* generating function, on the other hand, distinguishes between horizontal and vertical bonds. For a given bond animal B , we denote the number of horizontal (respectively vertical) bonds it contains by $|B|_{\leftrightarrow}$

(respectively $|B|_{\updownarrow}$). Generally speaking, it is not difficult to extend isotropic techniques (be they exact solutions or numerical expansions) to the anisotropic case, so one can obtain the anisotropic generating function without having to do too much more work.

Let \mathcal{G} be a set of bond animals on the square lattice, and let us count the elements of \mathcal{G} *anisotropically* by forming the *anisotropic* generating function:

$$gf(\mathcal{G}) = \sum_{Q \in \mathcal{G}} x^{|Q|_{\leftrightarrow}} y^{|Q|_{\updownarrow}} = \sum_{n,m=0}^{\infty} c_{n,m} x^n y^m,$$

where $c_{n,m}$ is the number of elements of \mathcal{G} containing exactly n horizontal bonds and m vertical bonds.² In the work that follows we write this generating function as a power series in y with coefficients that are series in x . Writing $\mathcal{G}_n = \{Q \in \mathcal{G}_n : |Q|_{\updownarrow} = n\}$ we have:

$$gf(\mathcal{G})(x, y) = \sum_{n=0} y^n \sum_{Q \in \mathcal{G}_n} x^{|Q|_{\leftrightarrow}} = \sum_{n=0} H_n(x) y^n.$$

The coefficient of y^n in the above generating function, $H_n(x)$, is the horizontal bond generating function of all animals in \mathcal{G} containing n vertical bonds (the set \mathcal{G}_n).

In some sense, the anisotropic generating function is a more manageable object than the isotropic. Splitting the set of animals \mathcal{G} , into separate simpler subsets, \mathcal{G}_n , gives us smaller pieces, each of which is easier to study than the whole. If one seeks to compute or even just understand the *isotropic* generating function then one must somehow examine *all* possible topologies or configurations³ that can occur in \mathcal{G} —it is perhaps for this reason that the only families of bond animals that have been solved are those with severe topological restrictions (such as column-convex polygons). On the other hand, if we examine the generating function of \mathcal{G}_n , then the number of different configurations that can occur is always finite. For example, consider self-avoiding polygons with $2n$ vertical bonds. If $n = 1$ all configurations are rectangles. If $n = 2$, then all configurations are vertically *and* horizontally convex, while if $n = 3$ all configurations are vertically *or* horizontally convex. The anisotropy allows one to study the effect that these configurations have on the generating function in a more controlled manner.

Similarly, instead of trying to study the properties of the whole (possibly unknown) generating function the anisotropy breaks the generating function into separate simpler pieces, $H_n(x)$, that can be calculated exactly for small n . By studying the properties of these coefficients, particularly their singularities, we can obtain some idea of the properties of the generating function as a whole.

² If we are considering a set of polygons, then the numbers of vertical and horizontal bonds are always even numbers, and so rather introducing extraneous factors of 2 we will enumerate families of polygons according to their horizontal and vertical *half-perimeters* (being exactly half the number of horizontal and vertical bonds). Rather than defining different notation for polygons we will simply take $|B|_{\leftrightarrow}$ and $|B|_{\updownarrow}$ to mean the horizontal and vertical half-perimeters when discussing a set of polygons.

³ We are being deliberately imprecise here—we will be more precise below.

1.3. Some examples

Let us consider the anisotropic generating functions of staircase polygons and self-avoiding polygons. However, before we do so, we need to define the family of polynomials that appear throughout the following work—in particular, we will show that the coefficients, $H_n(x)$, can be expressed in terms of rational functions of the form $x^k/(1-x^k)$, and so the factors of the denominators are *cyclotomic polynomials*.

Definition 1. The *cyclotomic polynomials*, $\Psi_k(x)$, are the factors of the polynomials $(1-x^n)$. In particular, $(1-x^n) = \prod_{k|n} \Psi_k(x)$.

The first few cyclotomic polynomials are underlined below.

$$\begin{aligned}\Psi_1: & (1-x) = \underline{(1-x)}, \\ \Psi_2: & (1-x^2) = (1-x)\underline{(1+x)}, \\ \Psi_3: & (1-x^3) = (1-x)\underline{(1+x+x^2)}, \\ \Psi_4: & (1-x^4) = (1-x)(1+x)\underline{(1+x^2)}, \\ \Psi_5: & (1-x^5) = (1-x)\underline{(1+x+x^2+x^3+x^4)}, \\ \Psi_6: & (1-x^6) = (1-x)(1+x)(1+x+x^2)\underline{(1-x+x^2)}.\end{aligned}$$

We call $\Psi_k(x)$ the k th cyclotomic polynomial, and say that its *order* is k . For any given integers $\{\alpha_i\}_{i \geq 1}$, we have the following factorisation

$$\prod_{n=1}^{\infty} (1-x^n)^{\alpha_n} = \prod_{n=1}^{\infty} \prod_{k=n/d} \Psi_k(x)^{\alpha_n} = \prod_{k=1}^{\infty} \Psi_k(x)^{\sum_{d=1}^{\infty} \alpha_{dk}}.$$

Example 3 (Staircase polygons). The anisotropic generating function of staircase polygons⁴ is known in closed form [16]:

$$P(x, y) = \frac{1}{2} \left(1 - x - y - \sqrt{(1-x-y)^2 - 4xy} \right).$$

Expanding $P(x, y)$ as a power series in y gives:

$$\begin{aligned}P(x, y) = & \frac{x}{1-x}y + \frac{x}{(1-x)^3}y^2 + \frac{x(1+x)}{(1-x)^5}y^3 + \frac{x(1+3x+x^2)}{(1-x)^7}y^4 \\ & + \frac{x(1+6x+6x^2+x^3)}{(1-x)^9}y^5 + \dots.\end{aligned}$$

⁴ Since this is a family of polygons the generating function enumerates anisotropic *half-perimeter*; the coefficient of $x^m y^n$ is the number of staircase polygons with $2m$ horizontal bonds and $2n$ vertical bonds.

The coefficients, $H_n(x)$, have the following properties:

- $H_n(x)$ is a rational function of x ;
- the degree of the numerator of $H_n(x)$ is $(n - 1)$, for all $n \geq 2$;
- the denominator of $H_n(x)$ is $(1 - x)^{2n-1}$;
- the coefficients of the numerators are positive, symmetric and unimodal.

Many other polygons models for which a closed form solution is known (such as the subsets of column-convex polygons) have similar properties (though the symmetry of numerator coefficients is lost in some cases) [4].

Similar properties have also been observed in three choice polygons and staircase polygons with a single staircase hole [4]—for these models no closed form solution exists, but their series expansion can be computed in polynomial time [6].

Example 4 (Self-avoiding polygons). The generating function of self-avoiding polygons, $P(x, y)$, remains elusive, and so its coefficients must be computed either by brute-force or the finite-lattice method. Expanding $P(x, y)$ as a power series in y , one observes [13] that the coefficients, $H_n(x)$, (which have been computed up to order $n = 14$) have the following properties:

- $H_n(x)$ is a rational function of x ;
- the degree of the numerator of $H_n(x)$ is equal to the degree of its denominator;
- the coefficients of the numerators are positive and unimodal, but not symmetric;
- if we write the denominator of $H_n(x)$ as $D_n(x)$, then the first ten are:

$$D_1(x) = (1 - x),$$

$$D_2(x) = (1 - x)^3,$$

$$D_3(x) = (1 - x)^5,$$

$$D_4(x) = (1 - x)^7,$$

$$D_5(x) = (1 - x)^9(1 + x)^2,$$

$$D_6(x) = (1 - x)^{11}(1 + x)^4,$$

$$D_7(x) = (1 - x)^{13}(1 + x)^6(1 + x + x^2),$$

$$D_8(x) = (1 - x)^{15}(1 + x)^8(1 + x + x^2)^3,$$

$$D_9(x) = (1 - x)^{17}(1 + x)^{10}(1 + x + x^2)^5,$$

$$D_{10}(x) = (1 - x)^{19}(1 + x)^{12}(1 + x + x^2)^7(1 + x^2).$$

This suggests that a new cyclotomic factor enters every third coefficient. Further, it enters with exponent 1, and increases by 2 in each subsequent coefficient (with the exception of $(1 + x)$ which has exponent 1 less than this pattern predicts).

Many other unsolved animal models (such as directed bond animals and lattice trees) display similar properties [10,11,13,14].

The denominator structure of self-avoiding polygons is starkly different to that of the previous (solvable) example. The denominators of staircase polygons contain only a single cyclotomic factor, $(1 - x)$, and so $H_n(x)$ contains only a finite number of poles as $n \rightarrow \infty$. On the other hand, if one extrapolates from the observed pattern of self-avoiding polygon denominators, then every cyclotomic factor will appear eventually, and so $H_n(x)$ has a dense set of poles on $|x| = 1$ as $n \rightarrow \infty$.

The observation of this demarcation between the properties of solved and unsolved families of bond animals led Guttmann and Enting [11] to propose it as a (numerical) test of the *solvability* of a model.⁵ In particular, if a function, $P(x, y)$, has coefficients whose poles form a dense set on $|x| = 1$, then the function is not a member of the most common functions of mathematics and physics—*differentiably-finite* functions (see [17,24]). In contrast, the generating functions of most solved models are differentiably-finite. The techniques described in this paper form the basis of a proof that the anisotropic generating function of self-avoiding polygons is not D-finite—this will be discussed in another paper [21] (see also [22]); this result is also being extended to other models [19,20].

The remainder of this paper is concerned with proving some of these observed properties of the coefficients, $H_n(x)$.

In Section 2 we develop a technique, which we call *haruspicy*, that shows how the set of bond animals may be partitioned into equivalence classes, so that each class has a simple rational generating function whose singularities are related to the horizontal bond configurations in the elements of the class. As a direct consequence of this one can show that $H_n(x)$ is rational, that the degree of its numerator is at most that of its denominator, and that its denominator is a product of cyclotomic polynomials.

In Section 3 we demonstrate how particular configurations of horizontal bonds give rise to the cyclotomic factors in the denominators of the coefficients of the anisotropic generating function, and then apply this result to a number of solved and unsolved models.

In Section 4 we prove a multiplicative upper bound for the denominator of $H_n(x)$ for a wide range of families of bond animals—that is, we find a sequence of polynomials, $\{B_n(x)\}_{n \geq 0}$, such that the denominator of $H_n(x)$ divides $B_n(x)$. This upper bound may be used to greatly reduce the amount of computation required in computer-aided expansions of anisotropic generating functions—specifically it shows that $H_n(x)$ may be computed exactly from the first $O(n^3)$ terms of its expansion.

⁵ Similar patterns have also been observed in the thermodynamic functions of the Ising and Potts models [10,11]. Such functions can be interpreted as enumerating families of graphs on the square lattice with complicated weights; the weights can be negative and the graphs can be disconnected. We hope to extend the techniques described in this paper to these problems.

2. Haruspicy

The technique we shall describe below allows us to determine properties of generating functions; both those known in closed form and those for which exponential time algorithms remain the best approach. This technique works by examining of horizontal bond configurations within specially chosen lattice animals and so we have used the word “*haruspicy*” to describe it. Haruspicy (pronounced “hə’rʌspɪsi”) is the art of divination by the examination of the forms and shapes of the organs of animals [5], and so seemed an appropriate term.

Let us start by considering the set of all self-avoiding polygons containing 2 vertical bonds; this is simply the set of all rectangles of height 1, and its horizontal half-perimeter generating function is $x/(1 - x)$ (see Fig. 3).

The smallest polygon (or the minimal polygon) in this set is the unit square. We can then obtain the other polygons from the unit square by “stretching” or “growing” the horizontal bonds (see Fig. 4). The unit square has generating function simply given by x , stretching the horizontal bonds to length n gives an n by 1 rectangle that contributes x^n to the generating function. Summing over all possible “stretches” gives $\sum_{n \geq 1} x^n = x/(1 - x)$ as required.

By reversing the stretching process, we can think of squashing the rectangles into shorter and shorter rectangles until we reach the unit square. This squashing process gives a (total) order on this set. The smallest element of this set under this order is the unit square. This idea can be extended to other animals, and we will introduce two different ways of “squashing” general animals. By examining the contents of these “squashed” animals and “stretching” them we can determine certain properties of anisotropic generating functions.

2.1. Columns, sections and partial orders

Definition 2. We will define a *column* of a given animal to be the horizontal bonds within a single horizontal lattice spacing of the animal. See Fig. 5. If the column contains k horizontal bonds we say it is a k -column. The number of k -columns in an animal, A , is denoted by $\gamma_k(A)$.



Fig. 3. Self-avoiding polygons containing exactly 2 vertical bonds. The horizontal half-perimeter generating function of this set is $x/(1 - x)$.

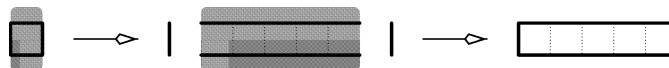


Fig. 4. Stretching or growing the horizontal bonds of the unit square will give any self-avoiding polygon with 2 vertical bonds.

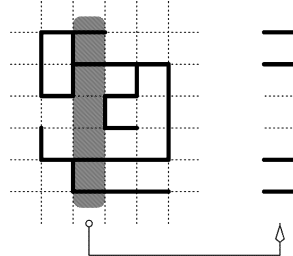


Fig. 5. A 4-column of a bond animal.

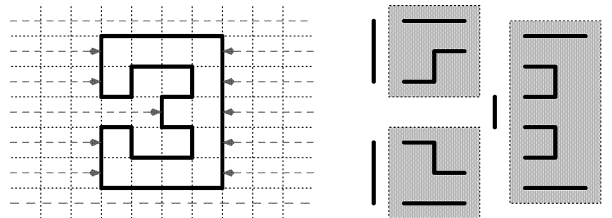


Fig. 6. *Section lines* (the heavy dashed lines) split the animal (in this example it is a polygon) into *pages*. Each column in a page is a *section*. This polygon is split into 3 pages, each containing 2 sections. 10 vertical bonds lie between pages and 4 vertical bonds lie within the pages.

Definition 3. We construct the *section lines* of an animal in the following way. Draw horizontal lines from the extreme left and the extreme right of the lattice towards the animal so that the lines run through the middle of each lattice cell. The lines are terminated when they first touch a vertical bond (see Fig. 6).

Cut the lattice along each section line from infinity until it terminates at a vertical bond. Then from this vertical bond cut vertically in both directions until another section line is reached. In this way the animal is split into *pages* (see Fig. 6); we consider the vertical bonds along these vertical cuts to lie *between* pages, while the other vertical bonds lie *within* the pages.

We call a *section* the set of horizontal bonds within a single column of a given page. Equivalently, it is the set of horizontal bonds of a column of an animal between two neighbouring section lines. A section with k horizontal bonds is a k -section. The number of k -sections in an animal, A , is denoted by $\sigma_k(A)$.

Definition 4. We say that a column is a *duplicate column* if the column immediately on its left (without loss of generality) is identical and there are no vertical bonds between them (see Fig. 7). We similarly define a *duplicate section*.

One can squash or reduce animals by *deletion* of duplicate columns by slicing the animal on either side of the duplicate column, removing the column and recombining the animal, as illustrated in Fig. 7. By reversing the column deletion process we define *duplication* of a column. We define *section-deletion* and *section-duplication* in an analogous manner.

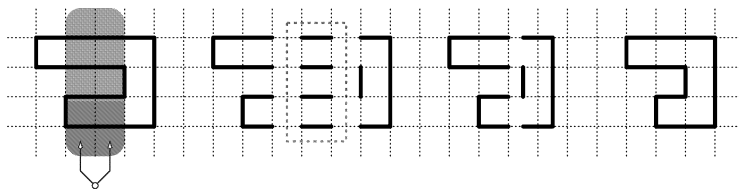


Fig. 7. Surgery. The process of column deletion. The two indicated columns are identical. Slice either side of the duplicate and separate the polygon into three pieces. The middle piece, being the duplicate, is removed and the remainder of the polygon is recombined. Reversing the steps leads to column duplication.

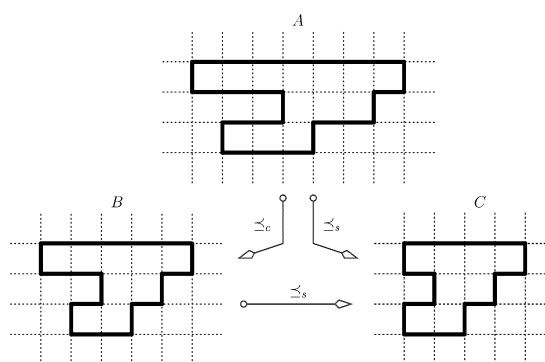


Fig. 8. Polygon A is reduced by a sequence of column deletions to polygon B (which is column-minimal) and a sequence of section deletions to polygon C (which is section-minimal). B can be reduced to C by section deletions, and hence is not section-minimal.

Using column- and section-deletion we can define two relations, \preceq_c and \preceq_s , on the set \mathcal{G}_n of animals with n vertical bonds.

Definition 5. For any two animals $P, Q \in \mathcal{G}_n$, we define the binary relations \preceq_c and \preceq_s by stating that:

- $P \preceq_c Q$ if $P = Q$ or P can be obtained from Q by a sequence of column-deletions, and
- $P \preceq_s Q$ if $P = Q$ or P can be obtained from Q by a sequence of section-deletions.

See Fig. 8 for example.

From this definition we immediately obtain the following lemma

Lemma 1. *The binary relations \preceq_c and \preceq_s are partial orders on the set of animals.*

Proof. Let A, B , and C be animals. A partial order must be reflexive, anti-symmetric and transitive. We state the proof for \preceq_c —the proof for \preceq_s is identical.

Reflexive. By definition $A \preceq_c A$.

Anti-symmetric. If $A \preceq_c B$, then either $A = B$ or $|A|_{\Leftrightarrow} < |B|_{\Leftrightarrow}$. Similarly if $B \preceq_c A$ then either $A = B$ or $|A|_{\Leftrightarrow} > |B|_{\Leftrightarrow}$. Hence if $A \preceq_c B$ and $B \preceq_c A$ then $A = B$.

Transitive. If $A \preceq_c B$ then there exists a sequence of column-deletions that takes B to A . Similarly if $B \preceq_c C$, then there exists another sequence of column-deletions that takes C to B . Concatenating these gives a sequence of column-deletions that takes C to A , and hence $A \preceq_c C$. \square

The first fact we can establish about these partial orders is to show that one implies the other:

Lemma 2. *Consider two bond animals A and B then*

$$A \preceq_c B \quad \Rightarrow \quad A \preceq_s B.$$

The converse is false.

Proof. Consider a duplicate column in B . The sections within this column must also be duplicate sections. The animal obtained by deleting column, can also be obtained by deleting the sections within the column. Hence if we obtain A from B by a sequence of column-deletions, then it can also be obtained by some sequence of section-deletions.

One can readily construct examples to show that the converse of this statement false. Consider animals B and C in Fig. 8. It is the case that $C \preceq_s B$, but $C \not\preceq_c B$. \square

2.2. Minimal animals and equivalence relations

If we take an animal and start to remove duplicate columns then we cannot reduce the animal to nothing. At some point we must reach an animal that contains no duplicate column. This animal we call a column-minimal animal. A little more formally we may write:

Definition 6. A *column-minimal animal*, A , is an animal such that for all animals B satisfying $B \preceq_c A$, then $B = A$. I.e., A cannot be reduced any further. We define a *section-minimal animal* in a similar way.

It is natural to ask that if A reduces to a column-minimal animal B and to a section-minimal animal C by some sequences of column- and section-deletions (respectively) then what is the relation between B and C ?

Lemma 3. *If an animal, P , is section-minimal then it is also column-minimal. The converse is false.*

Proof. Consider the contrapositive of this statement. If an animal, P , is not column-minimal, then there exists Q such that $Q \preceq_c P$. By Lemma 2, $Q \preceq_s P$, and so P is not section minimal. The animals depicted in Fig. 8, show that the converse of the lemma is false. \square

Consider an animal C that reduces to some column-minimal animal A by some sequence of column deletions. The next lemma tells us that A is in fact unique.

Lemma 4. *Every animal C reduces by column-deletions to a unique column-minimal animal. Similarly every animal reduces by section-deletions to a unique section-minimal animal. The column-minimal animal and section-minimal animal reached from C need not be the same.*

Proof. The columns of any animal, C , can be encoded (from left to right) as a sequence of columns, $(c_1^{\alpha_1}, c_2^{\alpha_2}, \dots, c_j^{\alpha_j})$, where $c_i^{\alpha_i}$ indicates α_i repetitions of the column c_i . Enforcing the additional constraint that $c_i \neq c_{i+1}$ will ensure the uniqueness of the α_i . Removing all duplicate columns will reduce C to some animal A , that is encoded by the sequence $(c_1^1, c_2^1, \dots, c_j^1)$. Clearly this is unique.

To prove the same result for section-deletion we note that section-deletion does not delete pages, nor does it move sections between pages, and so one can apply the above idea to the sections within each page of the animal. \square

Since every animal reduces to a unique minimal element by column deletion (or section deletion), the set of animals can be written as the disjoint union of posets, each of which contains a single minimal animal. Using this idea we can construct two equivalence relations on the set of animals:

Definition 7. We say that two animals, A and B , are *column-equivalent* if both A and B reduce to the same column-minimal animal. In this case we write $A \approx_c B$. Similarly we say that two animals, A and B , are *section-equivalent* if both A and B reduce to the same section-minimal animal. In this case we write $A \approx_s B$. See Fig. 9 for examples of column- and section-equivalence.

Lemma 5. *Column-equivalence and section-equivalence are equivalence relations.*

Proof. It follows almost directly from the definitions that column- and section-equivalence are reflexive, symmetric and transitive. \square

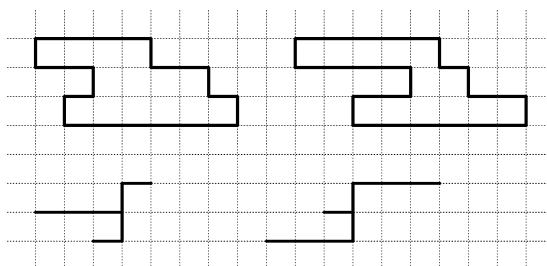


Fig. 9. The top two animals are column-equivalent (and so also section-equivalent), while the bottom two are section-equivalent but *not* column-equivalent.

Definition 8. Using the column- and section-equivalence relations, one can partition the set of animals into equivalence classes each of which can be characterised by the column-minimal (or section-minimal) animal within the class. We refer to the equivalence class of a column-minimal (respectively section-minimal) animal, A , as the *column-expansion* (respectively *section-expansion*) of A . We write:

$$\mathcal{X}_c(A) = \{B \in \mathcal{G} \mid A \preceq_c B\}, \quad (1)$$

$$\mathcal{X}_s(A) = \{B \in \mathcal{G} \mid A \preceq_s B\}. \quad (2)$$

Note that all the elements in such an expansions must have the same number of vertical bonds. We write the horizontal bond generating function of the expansion of a minimal element, A , as

$$G_c(A) = \sum_{B \in \mathcal{X}_c(A)} x^{|B|_{\leftrightarrow}} \quad \text{if } A \text{ is column-minimal}, \quad (3)$$

$$G_s(A) = \sum_{B \in \mathcal{X}_s(A)} x^{|B|_{\leftrightarrow}} \quad \text{if } A \text{ is section-minimal}. \quad (4)$$

Since \mathcal{G}_n is partitioned into equivalence classes, its generating function, $H_n(x)$, can be written as the sum of the generating function of each equivalence class.

Lemma 6. Let \mathcal{M}_c and \mathcal{M}_s be the sets of column-minimal animals and section-minimal animals (respectively) of \mathcal{G}_n , then

$$H_n(x) = \sum_{B \in \mathcal{G}_n} x^{|B|_{\leftrightarrow}} = \sum_{A \in \mathcal{M}_c} G_c(A) = \sum_{A \in \mathcal{M}_s} G_s(A).$$

Proof. Since each animal in \mathcal{G}_n is an element in the expansion of exactly one minimal animal the result follows. \square

Let us consider a set of bond animals, \mathcal{G}_n , the elements of which contain exactly n vertical bonds. We now ask how many equivalence classes (or minimal animals) are in \mathcal{G}_n . The exact number depends upon the family of animals under consideration, but we can show that it is finite.

Lemma 7. If \mathcal{G}_n is a set of animals with n vertical bonds, then the set of minimal elements in \mathcal{G}_n (w.r.t. either partial order) is finite.

Proof. By Lemma 3 every section-minimal animal is column minimal, so it suffices to prove the above lemma for column-minimal animals. We first show that all column-minimal animals with n vertical bonds have finite height and width.

Let P be a column-minimal animal in \mathcal{G}_n . Obviously P cannot contain more than n rows. Since there are no duplicate columns in P , between each pair of columns of P there

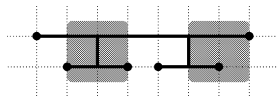


Fig. 10. Two pairs of consecutive columns are highlighted in this column-minimal animal. Between such consecutive columns there must be either a vertical bond (left) or a vertex of degree 1 (right).

must either be a vertical bond, or at least one horizontal bond must terminate, leaving a vertex of degree 1. See Fig. 10.

Let us bound the number of vertices of degree 1. If there are no vertical bonds, then there can be at most 2 vertices of degree 1. Each vertical bond can be attached to at most 4 lines of horizontal bonds. Hence there can be at most 4 vertices of degree 1 connected to each vertical bond (by lines of horizontal bonds). Consequently P can contain at most $4n + 2$ vertices of degree 1—if the vertical bonds are connected together, then this number will be lower, but we only require a rough bound. Between each pair of columns there must be either one of these vertices or a vertical bond, so the number of columns in P is bounded by $5n + 1$. We shall refine this in Section 4.

Hence every minimal animal in \mathcal{G}_n fits inside a box of height n and width $5n + 1$. Since there are only a finite number of bonds inside this box there can be only a finite number of column-minimal animals. \square

2.3. Dense families of animals and generating functions

Consider again the set of all self-avoiding polygons containing 2 vertical bonds. Clearly the column- and section-minimal animal is the unit square. The equivalence class of this animal is the original set, and so has generating function $x/(1 - x)$.

If we now consider the set of polygons having only 2 vertical bonds and odd horizontal half-perimeter, the minimal element is the same, but now the generating function is $x/(1 - x^2)$. Worse still is the subset of polygons which have 2 vertical bonds and *prime* horizontal half-perimeter; it still has the same minimal element, but has a very complicated generating function. To avoid these possibilities we restrict ourselves to *dense* families of animals.

Definition 9. A set of animals, \mathcal{G} is *dense* if it is closed under column- and section-deletion.

Most families of animals that are studied on the square lattice are dense, though some types of restricted self-avoiding walks [12,23] are not (e.g., the anti-spiral walk). Self-avoiding polygons on the *hexagonal lattice* are often considered (particularly for the purpose of computer aided enumeration) as polygons on the square lattice with the additional restriction that vertical bonds can only be placed according to a brick-work pattern—i.e., every second vertical edge is disallowed (see Fig. 11). Removing a duplicate column from such a polygon gives a polygon that violates the brick-work rule, and hence this family of animals is not dense. It should be possible to adapt the haruspicy techniques to animals on the brick-work lattice by requiring duplicate sections and columns be removed in pairs.

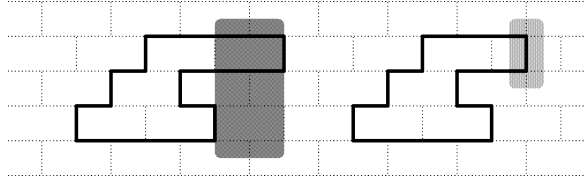


Fig. 11. The highlighted columns in the left hand polygon are duplicates. Removing one of the duplicate columns results in the polygon on the right which has a vertical bond where there is no edge on the brick-work lattice.

If we restrict ourselves to the examination of dense families of animals, then the generating functions of equivalence classes are simple rational functions.

Lemma 8. *If P is a column-minimal (respectively section-minimal) animal in a dense family of animals then its expansion has the following generating function:*

$$G_c(P) = \prod_k \left(\frac{x^k}{1-x^k} \right)^{\gamma_k(P)} \tag{5}$$

$$\left(\text{respectively } G_s(P) = \prod_k \left(\frac{x^k}{1-x^k} \right)^{\sigma_k(P)} \right). \tag{6}$$

Proof. We state the proof for column-minimal animals. Let P be a column-minimal animal; it can be encoded as a sequence of columns (c_1, \dots, c_j) , with $c_i \neq c_{i+1}$. Since P is part of a dense family of animals, given any $\alpha = (\alpha_1, \dots, \alpha_j) \in \mathbb{Z}^{+j}$ there exists an animal Q encoded by a sequence of columns $(c_1^{\alpha_1}, \dots, c_j^{\alpha_j})$.

So

$$\begin{aligned} \mathcal{X}_c(P) &= \bigcup_{\alpha} \{c_1^{\alpha_1}, \dots, c_j^{\alpha_j}\}, \\ G_c(P) &= \prod_i \sum_{\alpha_i} (x^{|c_i|_{\Leftrightarrow}})^{\alpha_i} = \prod_i \frac{x^{|c_i|_{\Leftrightarrow}}}{1-x^{|c_i|_{\Leftrightarrow}}}, \end{aligned} \tag{7}$$

where $|c_i|_{\Leftrightarrow}$ is the number of horizontal bonds in c_i . The result follows. The proof for $G_s(P)$ can be constructed in a similar way; instead of treating the animal as a whole, one considers the section configurations in each page in turn. \square

Directly from this we can deduce some of the properties of the coefficient of y^n in the anisotropic generating function of a dense set of animals:

Theorem 9. *If $P(x, y) = \sum_{n \geq 0} H_n(x)y^n$ is the anisotropic generating function of some dense family of animals, \mathcal{G} , then*

- $H_n(x)$ is a rational function;

- the degree of the numerator of $H_n(x)$ cannot be greater than the degree of its denominator; and
- the denominator of $H_n(x)$ is a product of cyclotomic polynomials.

Proof. From Lemma 6, $H_n(x)$ is the sum of the generating functions of the expansions of each of the minimal elements in \mathcal{G}_n . Lemmas 7 and 8 imply that this sum is a finite sum of rational functions with the desired properties. The result follows. \square

In the next section we further refine the above theorem to give more detailed information concerning the denominator factors, and then apply this to a number of models, both solved and unsolved.

3. Denominator factors and applications

3.1. The denominator of $H_n(x)$

We can sharpen Theorem 9 to determine which cyclotomic factors can appear in the denominator of $H_n(x)$ by noting that the denominator of $G_s(A)$ can only contain the cyclotomic factor $\Psi_k(x)$ if A contains a K -section, where K is some integer multiple of k .

Theorem 10 (Poles, columns, and sections). *If $H_n(x)$ has a denominator factor $\Psi_k(x)$, then \mathcal{G}_n must contain a column-minimal animal containing a K -column for some $K \in \mathbb{Z}^+$ divisible by k . Further if $H_n(x)$ has a denominator factor $\Psi_k(x)^\alpha$, then \mathcal{G}_n must contain a column-minimal animal that contains α columns that are K -columns for some (possibly different) $K \in \mathbb{Z}^+$ divisible by k .*

Similar results hold for k -sections and section-minimal animals.

Proof. The proof is identical for both partial orders. We state it here for section-minimal animals. Let $\mathcal{M} = \{M_i\}$ be the set of section-minimal animals $\in \mathcal{G}_n$.

$$\begin{aligned}
 H_n(x) &= \sum_i G_s(M_i) = \sum_i \prod_K \left(\frac{x^K}{1-x^K} \right)^{\sigma_K(M_i)} = \sum_i x^{|M_i|} \prod_k \Psi_k(x)^{-\sum_d \sigma_{kd}(M_i)} \\
 &= \frac{\langle \text{some polynomial in } x \rangle}{\prod_k \Psi_k(x)^{\mu_k}},
 \end{aligned}$$

where $\mu_k \leq \max_i \{\sum_d \sigma_{kd}(M_i)\}$ —this is an inequality since the numerator and denominator could share common cyclotomic factors. Consequently, if there is no minimal element M_i containing a K -section (for some K divisible by k) then $\mu_k = 0$, and the denominator cannot contain $\Psi_k(x)$.

Similarly, if for all $M_i \in \mathcal{M}$ the sum, $\sum_{d \geq 1} \sigma_{kd}(M_i) < \alpha$, (i.e., there is no minimal animal that contains α or more columns that are K -columns for K divisible by k) then $\mu_k < \alpha$. \square

Using the above theorem we can bound which cyclotomic factors may occur in the denominator of $H_n(x)$. More specifically, we can find a bound on the maximum order of cyclotomic factors appearing in the denominator of $H_n(x)$, by finding the highest order section or column appearing in a minimal animal in \mathcal{G}_n . The next lemma tells us which of the two partial orders will give the tighter bound.

Corollary 11 (Denominators with \preceq_s and \preceq_c). *Let \mathcal{G} be a dense family of animals, and let \mathcal{M}_c (respectively \mathcal{M}_s) be the set of column-minimal (respectively section-minimal) animals of \mathcal{G}_n . Let*

$$c = \max\{k \mid \exists A \in \mathcal{M}_c \text{ with } \gamma_k(A) > 0\} \quad \text{and}$$

$$s = \max\{k \mid \exists A \in \mathcal{M}_s \text{ with } \sigma_k(A) > 0\}.$$

Then $s \leq c$. Moreover, if Ψ_k is a factor of the denominator of $H_n(x)$ then $k \leq s$.

Proof. According to Theorem 10, if there is a factor of $\Psi_k(x)$ in the denominator of $H_n(x)$, then there must be a minimal animal that has a K -section (for some K divisible by k). Take a section-minimal animal, A , with an s -section. Since A is section-minimal, it is also column-minimal, and so contains a column with at least s horizontal bonds, so $s \leq c$. \square

If we can show that no animal in \mathcal{G}_n contains a column or section with more than k horizontal bonds, then Corollary 11 implies that the denominator of $H_n(x)$ can only contain cyclotomic factors of order $\leq k$. Further, it implies that if we wish to attempt to find such a bound, it is better to find the maximum number of horizontal bonds occurring in a section (the number s), rather than the maximum number of horizontal bonds in a column (the number c), since $s \leq c$, and gives tighter bounds on the order of the cyclotomic factors that can occur.

To illustrate Corollary 11, consider the polygons in Fig. 12 enumerated by their horizontal half-perimeters. One can see that

$$\mathcal{X}_s(P) = \mathcal{X}_c(P) \uplus \mathcal{X}_c(Q) \uplus \mathcal{X}_c(R)$$

and hence

$$G_s(P) = G_c(P) + G_c(Q) + G_c(R),$$

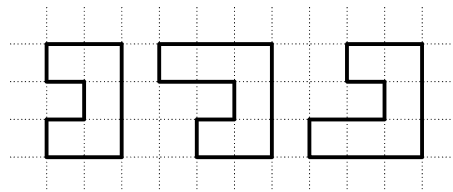


Fig. 12. Polygons P , Q , and R (from left to right) are all column-minimal, but Q and R reduce to P under \preceq_s . So while P contains only 1-sections, all contain 1 and 2 columns.

where

$$G_s(P) = \frac{x^3}{(1-x)^3}, \quad G_c(P) = \frac{x^3}{(1-x)^2(1+x)},$$

$$G_c(Q) = \frac{x^4}{(1-x)^3(1+x)}, \quad G_c(R) = \frac{x^4}{(1-x)^3(1+x)}.$$

So the column-minimal polygons suggest the existence of a higher order cyclotomic factor, $(1+x)$, which the section-minimal polygon does not. Summing over all minimal diagrams will give the same generating function, but there will be *more* cancellations using column-minimal animals.

3.2. Applications

Theorem 10 tells us that if there is no animal in \mathcal{G}_n that contains a k section (or a K -section for any K being an integer multiple of k), then the denominator of $H_n(x)$ does not contain a factor of $\Psi_k(x)$. Further, if there is no *minimal animal* that contains α k -sections in \mathcal{G}_n (or a total of α K -sections for any K being an integer multiple of k), then the denominator of $H_n(x)$ cannot contain a factor of $\Psi_k(x)^\alpha$. In the following corollary we apply this idea to a number of solved and unsolved families of animals.

Corollary 12. *We have the following results on the coefficient denominators in the anisotropic generating functions of various families of dense animals:*

- The coefficient of y^n in the anisotropic generating function of any subset of column convex polygons can only contain denominator factors $(1-x)$.
- The coefficient of y^n in the anisotropic generating function of any subset of row convex polygons can only contain denominator factors $(1-x)$.
- The coefficients of y^n in the anisotropic generating function of 3-choice polygons can only contain denominator factors $(1-x)$ and $(1+x)$.
- For any dense family of animals (such as bond animals or lattice trees) containing n vertical bonds, \mathcal{G}_n , the horizontal bond generating function $H_n(x)$, cannot contain the denominator factor $\Psi_k(x)$ if $n < 2k - 2$.
- The exponent of $\Psi_k(x)$ in the denominator of $H_{2k-2}(x)$ is at most k .

Proof. We claim that all of the above families of animals are dense, and this fact may be easily checked. We proceed by showing how many vertical bonds are required to construct an animal that contains a given number of k -sections, and then the results follow by application of Theorem 10.

Consider the polygons given in Fig. 13.

- Column convex polygons by definition can only have 2 horizontal bonds in each column and hence only contain 1-sections.
- Row convex polygons containing k rows, can have k -columns, but they are restricted to only have 1-sections due to row convexity.

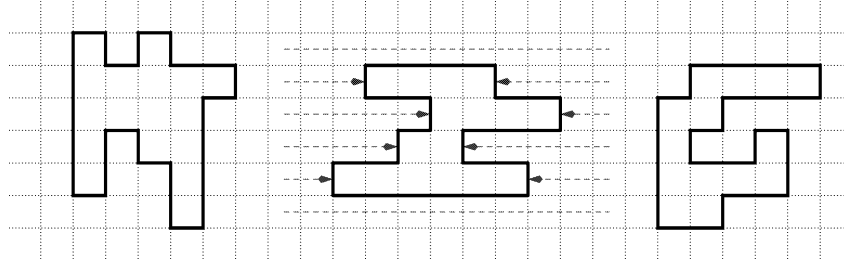


Fig. 13. A column convex polygon, a row-convex polygon and a 3-choice polygon.

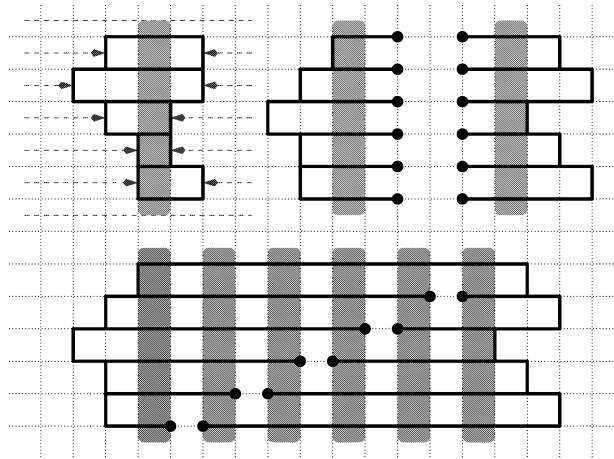


Fig. 14. A single 6-section requires 10 vertical bonds. Between the left of the rightmost 6-section and the right of the leftmost 6-section there can be at most 12 vertices of degree 1, but at least two of these are required to connect the left of the animal to its right.

- Each column of a 3-choice polygon can contain at most 4 horizontal bonds, and so all 3-choice polygons contain only 1- and 2-sections.

Consider the animals drawn in Fig. 14.

- To construct a k -section, at least $2k - 2$ section lines need to be blocked, each requiring a single vertical bond.
- An animal containing a k -section and exactly $2k - 2$ vertical bonds must be of height $k - 1$. In such an animal if two k -sections are in adjacent columns they must be identical, and so the animal is not section-minimal.

From the $k - 1$ vertical bonds on the left (and similarly on the right) there can be at most k lines of horizontal bonds towards the right (and left). One of these lines from the left must connect to one of the lines from the right, leaving $2k - 2$ lines of horizontal bonds that can terminate in a vertex of degree 1. Hence between the vertical bonds on

the left of the k -section and those on the right, there can be at most $2k - 2$ degree one vertices.

Between each pair of columns there must be at least one of these degree 1 vertices, so there can be a total of $2k - 1$ sections (between the vertical bonds). Not all of these can be k -sections, since if two k -sections are next to each other there will be no degree 1 vertex between them. So there can be k k -sections, with $k - 1$ not- k -sections between them. \square

The above corollary does not address the problem of finding a bound on the exponent of $\Psi_k(x)$ in the denominator of $H_n(x)$ for general n and k . This requires considerably more work and is the subject of the following section.

4. Bounding denominators and a weak half solution of bond animals

We are able to find a multiplicative upper bound for the denominator of general bond animals, by finding upper bounds for the exponents of its cyclotomic factors. In a future paper we will also do this for self-avoiding polygons (see also [22]). This multiplicative upper bound actually determines a little over half of the unknowns in the generating function and so can be considered, in some very weak sense, to give a little over half a solution of this problem.

4.1. Bounding denominators

The main result of this subsection is to prove the following multiplicative upper bound on the denominator of $H_n(x)$ (which we denote $D_n(x)$) in the anisotropic generating function of bond animals:

$$D_n(x) \left| \left(\Psi_1(x)^{3n+1} \prod_{k=2}^{\lfloor n/2 \rfloor + 1} \Psi_k(x)^{2n-3k+4} \right). \quad (8)$$

We obtain this bound by applying Theorem 10; the exponent of $\Psi_k(x)$ in $D_n(x)$ must be less than the maximum number of K -sections (where K is a multiple of k) that may occur in a section-minimal animal with n vertical bonds. We treat the cases $k = 1$ and $k \geq 2$ separately:

- The exponent of $\Psi_1(x)$ is bounded above by the maximum total number of sections in a section-minimal animal. We find this bound by first finding the number of pages, and then the number of sections that may lie in these pages. Maximising this number gives the bound.
- For $k \geq 2$ we use a similar idea, but it is complicated by two extra conditions. Firstly that a page containing a k -section must contain at least $k - 1$ rows and so not all pages may contain k -sections, and secondly that the maximum number of sections in a page is greater than the maximum number of k -sections in a page.

We start by finding the number of pages in an animal with a given numbers of rows with one vertical bond and a given number of rows with two or more vertical bonds.

Lemma 13. *The maximum number of pages in an animal is $(3R_1 + 4R_2 + 1)$, where R_1 is the number of rows containing a single vertical bond, and R_2 is the number of rows containing two or more vertical bonds. Consequently an animal with V vertical bonds can contain at most $(3V + 1)$ pages.*

Further $2(R_1 + R_2) + 2$ of these pages lie “outside” vertical bonds (i.e., only touching vertical bonds on one side), leaving $(R_1 + 2R_2 - 1)$ that lie between vertical bonds.

Proof. An animal with no vertical bonds contains a single page, while a single row animal contains 4 pages if it contains 1 vertical bond, or 5 pages otherwise (see Fig. 15). We proceed by appending rows to the animal (see Fig. 16).

By appending a row with 1 vertical bond, two new pages are created (to the left and right of the bond), while one existing page may be split in two—increasing the total number of pages by at most 3. Similarly by appending a row with 2 or more vertical bonds, two new pages are created (to the left and right of the vertical bonds that block the section lines), while two existing pages may be split in two—increasing the total number of pages by at most 4. Hence the total number of pages is at most $3R_1 + 4R_2 + 1$. We note that the total number of pages will be less than this if vertical bonds (blocking section lines) have the same horizontal ordinate. The total number of pages is maximised when no row contains more than a single vertical bond.



Fig. 15. The number of pages in a bond animal of height 0 or 1. We note that the rightmost animal contains four pages that lie “outside” the animal and only a single page lying between vertical bonds.

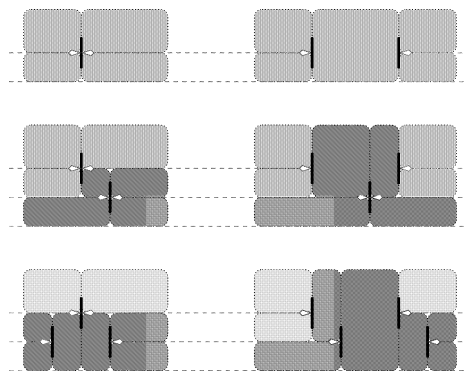


Fig. 16. Appending a row with a single vertical bond adds 2 new pages and splits an existing page (middle diagram). Appending a row with two or more vertical bonds, adds 2 new pages and splits 2 existing pages (bottom diagram). If the new vertical bonds have the same horizontal ordinate as those in the previous row, then less new pages will be created.

Between two (vertically) consecutive section-lines there is always a page to the extreme left of the animal, and another to the extreme right. The number of such pages equal to twice the number of rows in the animal plus two, i.e., $2(R_1 + R_2) + 2$. \square

Lemma 14. *A section-minimal animal with V vertical bonds contains at most $2(V + 1)$ vertices of degree 1 and at most $(3V + 1)$ sections. This bound is tight since we can construct a section-minimal animal with V vertical bonds and $(3V + 1)$ 1-sections.*

Proof. We prove this by determining the number of pages and the number of vertical bonds and vertices of degree 1 that may lie *inside* these pages, and so *between* sections.

- Assume that the animal contains R_1 rows containing 1 vertical bond and R_2 rows containing 2 or more vertical bonds, and hence contains at most $3R_1 + 4R_2 + 1$ pages.
- An animal with no vertical bonds is simply a horizontal line and so contains 2 vertices of degree 1. Consider constructing an animal by attaching new vertical bonds to an existing animal. Each new vertical bond can be connected to at most 4 vertices of degree 1, however in order to be connected to the rest of the animal, one of these vertices must be connected to another vertex of degree 1 on the animal. Hence adding a new vertical bond creates at most 2 new vertices of degree 1.
- Let the number of vertical bonds in this animal is $V = R_1 + 2R_2 + M$, of which at most M do not block section lines, and so may lie inside pages.
- Such an animal can be “completed” by appending horizontal bonds to its left and right (see Fig. 17) so that it has $2(R_1 + R_2 + 1)$ vertices of degree 1 lying to the extreme left or right of a row. Hence of the maximum possible $2V + 2 = 2R_1 + 4R_2 + 2M + 2$ vertices of degree 1, at most $2R_2 + 2M$ may lie between sections (not at the extreme end of a row). If there were an animal with more than this number, then by completing it one could obtain an animal with more than $2V + 2$ vertices of degree 1, giving a contradiction.
- Consider a page in a section-minimal animal. Between two sections in this page there must be either a vertical bond, or a vertex of degree 1 (otherwise there would be duplicate sections). Hence if a page contains c vertical bonds and d vertices of degree 1, then it can contain at most $c + d + 1$ sections.
- If a section-minimal animal contains n pages, with c_i vertical bonds and d_i vertices of degree 1 in page i , then it contains at most $\sum_{i=1}^n (c_i + d_i + 1)$ sections. Hence a section-minimal animal contains $3R_1 + 4R_2 + 1$ pages, with M vertical bonds and $2R_2 + 2M$ vertices of degree one lying inside these pages, then it can contain at most $3R_1 + 6R_2 + 3M + 1 = 3V + 1$ sections.

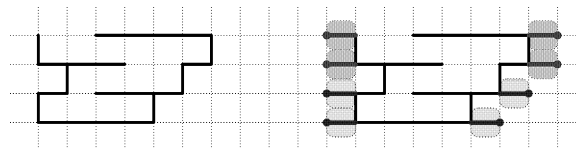


Fig. 17. Any section-minimal animal (left) can be “completed” (right) so that it has $2(R_1 + R_2 + 1)$ vertices of degree 1 lying to the extreme left or right of a row.

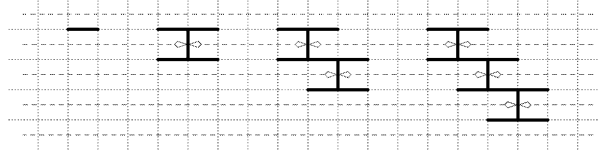


Fig. 18. By concatenating “I”-shaped configurations of bonds we obtain a section-minimal animal with V vertical bonds and $3V + 1$ sections. This shows that the bound obtained in Lemma 14 is tight.

- By concatenating “I”-shaped configurations of bonds (each containing a single vertical bond) we can construct a section-minimal animal with V vertical bonds and $3V + 1$ sections. This construction is shown in Fig. 18. \square

These lemmas are sufficient to bound the exponent of $\Psi_1(x)$ in the denominator of $D_n(x)$. We now move onto the analogous result for general $\Psi_k(x)$. Since a k -section contains at least $(k - 1)$ rows, we need to find the number of pages that contain at least $(k - 1)$ rows, and hence may contain k -sections.

We determine the number of pages that contain at least $(k - 1)$ rows, by considering how many pages lie to the left (or right) of the leftmost (or rightmost) such page.

Lemma 15. *Consider a section-minimal animal, A , that contains a page of height $h \geq 1$. To the left (right) of the leftmost (rightmost) such page, there must be at least h vertical bonds.*

Further it is always possible to construct a second animal, B , from A , such that there are $h - 1$ pages lying between it and the h vertical bonds to its left (right). This does not alter the total number of vertical bonds, nor the number of rows with one vertical bond, nor the number of rows with two or more vertical bonds.

Proof. Without loss of generality, let us consider the pages and bonds to the left of the leftmost page of height h . Let us denote the page by P . Since every section-line must be blocked, the number of vertical bonds to the left of P must be equal to its height (this was described in the proof of Corollary 12).

See Fig. 19. Remove all of the animal lying in the rows to the left of P . Since this portion of the animal is bounded above and below by section-lines, this can be done without changing the rest of the animal. We now replace the deleted part of the animal with a new configuration of bonds that contains $h - 1$ pages, that preserves connectivity, minimality, the total number of vertical bonds, the number of rows with 1 vertical bond, and the number of rows with two or more vertical bonds.

The configuration of bonds we add is a staircase-like configuration and is illustrated in Fig. 20. This configuration contains $h - 1$ pages, ensures connectivity and preserves the number of rows with two or more vertical bonds. The number of rows with one vertical bond is also conserved since each row incident on P must contain at least two vertical bonds.

To conserve the total number of vertical bonds we append a sequence of cells to the bottom row of the staircase (see rightmost illustration in Fig. 20) the total number of

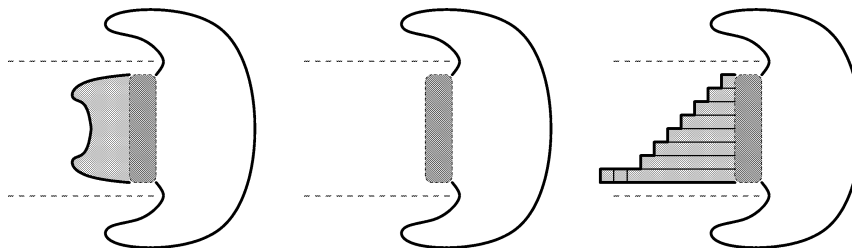


Fig. 19. If an animal contains a page of height h , then one can construct a second animal that has $h - 1$ pages to the left of the left-most such page. Start by deleting all of the animal to the left of the leftmost page of height h . Then attach a staircase configuration (as described in the proof of Lemma 15 and Fig. 20) to the left of the page of height h .

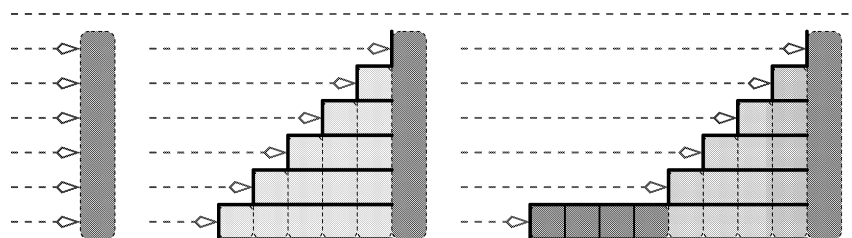


Fig. 20. Constructing a “staircase” to be attached to a page of height h . There are $h - 1$ pages between the vertical bonds and the page of height h . In order to conserve the number of vertical bonds one can attach a row of cells (as shown in the rightmost figure). All of these cells lie in the page of height 1 and so do not alter the number of pages.

vertical bonds can be conserved and at the same time the number of pages will remain $h - 1$. The sections in the page of height 1 are separated by vertical bonds, while every other page in the staircase contains only a single section, and so minimality is conserved. \square

Using the above lemma we are able to determine how many pages in an animal have height at least $(k - 1)$ and so may contain k -sections. We are also able to determine the maximum number of vertical bonds and vertices of degree 1 lying in these pages. These facts will be used to determine the maximum number of k -sections in a section-minimal animal.

Lemma 16. *Since an animal containing a k -section must have at least $(k - 1)$ rows with two or more vertical bonds, consider a section minimal animal which contains R_1 rows with a single vertical bond, $(k - 1 + R_2)$ rows with 2 or more vertical bonds and a total of $(R_1 + 2k - 2 + R_2 + M)$ vertical bonds. Then*

- *this animal contains at most $(R_1 + 2R_2 + 1)$ pages that may contain k -sections, and*
- *these pages contain at most $2(R_2 + M + k - 1)$ vertices of degree 1, and M vertical bonds.*

Proof.

- By Lemma 13, such an animal contains at most $3R_1 + 4(k - 1) + 4R_2 + 1$ pages, of which at most $R_1 + 2(k - 1) + 2R_2 - 1$ lie between vertical bonds and so may contain k -sections.
- Of this maximum of $R_1 + 2k + 2R_2 - 3$ pages, no animal may have more than $R_1 + 2R_2 + 1$ pages of height $\geq (k - 1)$. If there were such an animal, then by Lemma 15 one can construct a new animal from this one such that it has at least $(k - 2)$ pages to the left of the leftmost such page, and similarly to the right of the rightmost page. This would give a total of more than $R_1 + 2(k - 1) + 2R_2 - 1$, contradicting Lemma 13.
- By Lemma 14, this animal contains at most $2R_1 + 4(k - 1) + 4R_2 + 2M + 2$ vertices of degree 1, of which $2(R_1 + R_2 + k)$ lie outside vertical bonds, leaving $2(k + R_2 + M - 1)$ that may lie inside the pages containing k -sections.
- $R_1 + 2(k - 1) + 2R_2$ vertical bonds must block section-lines, leaving at most M vertical bonds that may lie inside pages containing k -sections. \square .

Using the above lemma, one may determine how many sections lie within pages that can contain k -sections (being those that contain at least $(k - 1)$ rows). One can then obtain an upper bound on the number of k -sections, by assuming that all of these sections are k -sections. A much sharper result, however, may be obtained by noting that in a page containing a given number of vertical bonds and vertices of degree 1, the maximum number of k -sections is less than the maximum number of sections.

Lemma 17. *A page in a section-minimal animal that contains c vertical bonds and d vertices of degree 1, may contain at most $(c + \lfloor d/2 \rfloor + 1)$ k -sections.*

Proof. Two consecutive k -sections in a page of a section-minimal animal, may not be duplicates, and so must be separated either by a vertical bond, some number of sections (that are not k -sections) or some number of vertices of degree 1.

- If there are one or more vertical bonds lying between the k -sections, then they will not be duplicates.
- If there is some number of sections lying between the two k -sections, then all of these sections must be separated by either a vertical bond, or a vertex of degree 1. Hence between the two k -sections there must be at least two vertical bonds, two vertices of degree 1 or one vertical bond and one vertex of degree 1.
- The k -sections may not be separated by a single vertex of degree 1, since they would then have different numbers of horizontal bonds and so could not both be k -sections. Hence there must be at least two vertices of degree 1 between them.

Consequently, between two k -sections in a page, there must either be at least a single vertical bond, two vertices of degree 1, or a vertical bond and a vertex of degree 1. Hence the maximum number of k sections in a page is $a + \lfloor b/2 \rfloor + 1$. \square

If we consider the total number of K -sections (where K is any integer multiple of k) in a page containing c vertical bond and d vertices of degree 1, then the above lemma still holds *except* when $k = 1$, since two such sections need not have the same number of horizontal bonds. For example, a 2-section and a 1-section may be separated by a single vertex of degree 1. This is the reason for the different bounds obtained when $k = 1$ and when $k \geq 2$.

We are now in a position to prove an upper bound on the number of k -sections appearing in a section-minimal animal with a given number of vertical bonds.

Theorem 18. *For fixed $k \geq 2$, consider a section-minimal animal with $V = (2k - 2 + R_1 + 2R_2 + M)$ vertical bonds, and R_1 rows containing a single vertical bond and $(R_2 + k - 1)$ rows containing at least 2 vertical bonds. This animal contains at most $(k + R_1 + 3R_2 + 2M)$ k -sections.*

For fixed $V \geq 2(k - 1)$ and $k \geq 2$ the number of k -sections is maximised when the animal has the minimum number of rows—i.e., $R_1 = R_2 = 0$, and $M = (V - 2k + 2)$. The maximum number of k -sections in a section-minimal animal with $V \geq (2k - 2)$ vertical bonds is $(2V - 3k + 4)$. This bound is tight.

Proof.

- Since there are at most $R_1 + 2R_2 + 1$ pages that can contain k -sections, and these pages contain at most M vertical bonds and at most $2(R_2 + k + M - 1)$ vertices of degree 1 in these pages, there can be at most $M + (R_2 + k + M - 1) + (R_1 + 2R_2 + 1) = 2M + 3R_2 + k + R_1$ k -sections.
- Maximising $2M + 3R_2 + k + R_1$ for fixed k and fixed $V = 2(k - 1) + R_1 + 2R_2 + M$, is equivalent to maximising $2M + 3R_2 + R_1$ on the surface $R_1 + 2R_2 + M = \text{constant}$, where $R_1, R_2, M \geq 0$. This maximum occurs when $R_1 = R_2 = 0$.
- When $R_1 = R_2 = 0$, then $M = V - 2(k - 1)$, and the number of k -section is $2M + k = 2V - 3k + 4$.
- In Corollary 12 it was shown how a section-minimal animal with $2(k - 1)$ vertical bonds and k k -sections could be constructed. In Fig. 21 a construction is given of a section-minimal animal containing $k + 2M$ k -sections and $V = 2(k - 1) + M$ vertical bonds. □

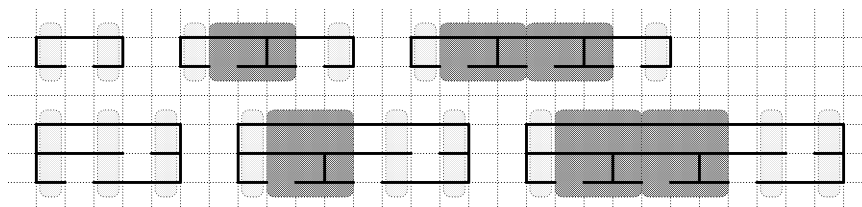


Fig. 21. An animal with 2 2-sections, and another with 3 3-sections. Introducing the highlighted configuration of bonds increases the number of vertical bonds by 1 and the number of k -sections by 2.

In order to apply Theorem 10 to the above result and obtain an upper bound on the exponent of $\Psi_k(x)$ in a given denominator, we must take into account the fact that a k -section, a $2k$ -section and so on, all contribute a factor of $\Psi_k(x)$. In order to obtain a correct bound on the denominator we need a bound on the number of k -, $2k$ -, ... sections in an animal with V vertical bonds.

As noted above Lemma 17 does break down when considering the maximum such sections for $k = 1$, since a 2-section and a 1-section can be separated by a single vertex of degree 1, and both contribute a factor of $\Psi_1(x)$. Fortunately the other lemmas required to prove the above theorem are not changed by considering the maximum number of such sections.

Theorem 19. *A section-minimal animal with V vertical bonds contains at most $(3V + 1)$ 1-sections. For $k \geq 2$, a section-minimal animal with $V \geq 2(k - 1)$ vertical bonds contains at most $(2V - 3k + 4)$ k -sections. Consequently the exponents of cyclotomic factors appearing in the denominators of $H_n(x)$ are bounded above according to:*

- the factor $\Psi_1(x)$ in the denominator of $H_n(x)$ appears with an exponent of at most $3n + 1$, and
- the factor, $\Psi_k(x)$, in the denominator of $H_n(x)$ appears with an exponent of at most $2n - 3k + 4$ (for $k \geq 2$).

Hence the denominator of $H_n(x)$ is bounded above (multiplicatively) by

$$D_n(x) \left| \left(\Psi_1(x)^{3n+1} \prod_{k=2}^{\lfloor n/2 \rfloor + 1} \Psi_k(x)^{2n-3k+4} \right). \right. \quad (9)$$

Proof. Apply Theorem 10 to Lemma 14 and Theorem 18. \square

Corollary 20. *The denominator of $H_n(x)$ in the generating function of any dense subset of bond animals (i.e., closed under section deletion and duplication) is also bounded above multiplicatively by*

$$D_n(x) \left| \left(\Psi_1(x)^{3n+1} \prod_{k=2}^{\lfloor n/2 \rfloor + 1} \Psi_k(x)^{2n-3k+4} \right). \right. \quad (10)$$

Proof. If the denominator is not bounded by this expression, then there must be a k and V such that there exists a section-minimal animal with V vertical bonds that contains more than $2V - 3k + 4$ k -sections in this dense subset of animals. However this section-minimal animal would also be section-minimal in the set of all bond animals which contradicts Theorem 18. \square

Where such data exists, one can compare numerical expansions with the above bound, and in the case of bond animals and lattice trees [13] it appears to be tight (for all the available coefficients of y). For other models, such as self-avoiding polygons and

directed bond animals, the bound is not particularly good, and finding better bounds for these models will be the subject of future papers [20,21].

4.2. A weak half solution, and easier computation

To recap, the anisotropic generating function can be written as

$$P(x, y) = \sum_{n \geq 0} H_n(x) y^n = \sum_{n \geq 0} \frac{N_n(x)}{D_n(x)} y^n,$$

where the degree of the numerator is no greater than that of the denominator. The above multiplicative upper bound determines the denominators of these coefficients up to some product of cyclotomic factors, and hence also gives a bound on the degree of numerators.

We can obtain a rough asymptotic estimate of the degree of the numerator and denominator as a function of n by examining the asymptotics of the degree of $\Psi_k(x)$, which is given by the Euler totient function $\phi(k)$. It is known [18] that $\sum_{k=1}^n \phi(k)/k \sim 6n/\pi^2$, and so the “average” $\langle \phi(k)/k \rangle$ is approximately $6/\pi^2$. Using this one can show that the degree of the n th denominator is asymptotic to $3n^3/(4\pi^2)$.

Thus there are approximately $\sim 3n^3/(2\pi^2)$ unknowns in the n th coefficient of the asymptotic expansion of the generating function. The bound on the denominator determines half of these, leaving $\sim 3n^3/(4\pi^2)$ that must be determined. In this way, it can be considered in some *very* weak sense to be half a solution of the bond animal and lattice tree problems.

From a computational point of view, the haruspicy technique greatly reduces the amount of work required to compute anisotropic generating functions. It reaffirms (the well observed) fact that the coefficients of y are rational functions of x , and more importantly, it bounds the degrees of the numerator and denominator and so bounds the order of the expansion necessary to fully determine each of these coefficients. Most importantly, it determines a multiplicative upper bound for the denominator which appears to be quite tight, and so reduces the number of unknowns (and hence the required expansion order) by a factor of two.

In problems possessing horizontal-vertical symmetry, still more unknowns can be determined using the $x \leftrightarrow y$ symmetry of the generating function. In particular, if we know the first $(n - 1)$ coefficients of y in the asymptotic expansion, then we know the coefficients of $x^n y^i$ for $0 \leq i < n$ and hence the first n coefficients of x in the expansion of the coefficient of y^n . These coefficients can be used to determine n unknowns in the numerator polynomial N_n . See [4] for details of this procedure.

A different but related method, has been used to find anisotropic generating functions (see [4] for example). It works by using certain spatial and functional symmetries of a problem, together with a knowledge of the denominators of its coefficients, to determine some or all the unknowns in the numerators. Such techniques have been used in combinatorics and statistical mechanics, and in certain circumstances yield full and elegant solutions. Unfortunately it does not appear that they can be applied here.

We also note that it is possible to demonstrate the rationality of $H_n(x)$ by a transfer matrix method (see [7–9] for example), however the dimensions of the matrix grow

exponentially with n , and so give a bound on the degree of the numerators and denominators that is exponential in n , rather than the polynomial bound given here.

5. Conclusions

We have developed new techniques for the examination of anisotropic generating functions. These techniques allow us to prove a number of observed properties of the coefficients of bond animal generating functions. It also allows us to find upper bounds for the exponents of cyclotomic factors appearing in the denominators of these coefficients, and thus greatly reduce the amount of computation required to obtain anisotropic series expansions.

We will apply these techniques to other bond animal models, to other families of bond animals. In particular, it is possible to tighten the bound given in Corollary 20 for specific families; such results for self-avoiding polygons are currently in preparation [21] and we are extending to directed bond animals [20].

Perhaps the most interesting extension of this technique is to prove that (in certain cases) there is no cancellation of cyclotomic factors between the numerator and denominator of $H_n(x)$, as is suggested by the apparent tightness of the denominator bound. In particular, it is possible [21,22] to prove a lower bound for the exponent of $\Psi_k(x)$ in the denominators of certain coefficients of the self-avoiding polygon generating function. An important corollary of this result is that this generating function is not differentiably-finite, and hence distinctly different from almost all solved families of animals.

We also hope that it will be possible to apply some of these techniques to problems in lattice statistical mechanics, such as Ising-type models of magnets. The thermodynamic functions in these models can be interpreted as generating functions of bond animals with complicated weights [25].

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