Haruspicy 3: The anisotropic generating function of directed bond-animals is not D-finite

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Abstract

While directed site-animals have been solved on several lattices, directed bond-animals remain unsolved on any nontrivial lattice. In this paper we demonstrate that the anisotropic generating function of directed bond-animals on the square lattice is fundamentally different from that of directed site-animals in that it is not differentiably finite. We also extend this result to directed bond-animals on hypercubic lattices. This indicates that directed bond-animals are unlikely to be solved by similar methods to those used in the solution of directed site-animals. It also implies that a solution cannot be conjectured using computer packages such as GFUN [A MAPLE package developed by B. Salvy, P. Zimmermann, E. Murray at INRIA, France, available from http://algo.inria.fr/libraries/ at time of submission; B. Salvy, P. Zimmermann, GFUN: A MAPLE package for the manipulation of generating and holonomic functions in one variable, ACM Trans. Math. Software 20 (2) (1994) 163–177] or differential approximants [A.J. Guttmann, Asymptotic analysis of coefficients, in: C. Domb, J. Lebowitz (Eds.), Phase Transit. Crit. Phenom., vol. 13, Academic Press, London, 1989, pp. 1–234, programs available from http://www.ms.unimelb.edu.au/~tonyg].
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1. Introduction

The enumeration of lattice animals is a long-standing problem in enumerative combinatorics and finds applications in statistical physics and theoretical chemistry. Though the subject has received considerable attention over many years, the problem remains unsolved.

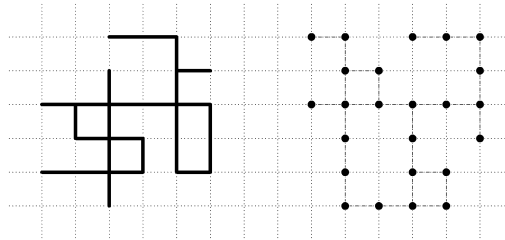*E-mail address:* a.rechnitzer@ms.unimelb.edu.au.
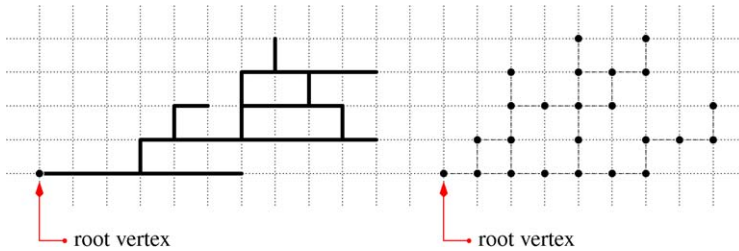
Fig. 1. A bond-animal and a site-animal.

Fig. 2. A directed bond-animal and a directed site-animal.

Definition 1. A *bond-animal* is a connected union of bonds (edges) on a lattice.[1] Similarly a *site-animal* is a connected union of sites (vertices). Two animals are considered to be the same if they are translates of each other. See Fig. 1.

In spite of the difficulty of enumerating general lattice animals, many subclasses have been solved. In almost all cases it has only been possible to count animals with quite severe topological restrictions—such as directedness or convexity. In this paper we focus on directed animals.

Definition 2. A bond-animal is *directed* if it contains a special vertex called the *root vertex* such that all bonds in the animal may be reached from the root vertex by paths that take only north and east steps and are contained within the animal. Similarly a site-animal is directed if it contains a root vertex and all other sites can be reached from it by taking only north and east steps. See Fig. 2.

Directed site-animals were first solved around 20 years ago by Dhar [7,8] by mapping the problem to a hard-core lattice gas, and then subsequently by a number of authors using more geometric and bijective methods (such as [2,3,9,15]). The resulting generating function is a simple algebraic function:

$$S(q) = \sum_{\substack{A \in \\ \text{directed} \\ \text{site-animals}}} q^{|A|} = \frac{1}{2}\left(\sqrt{\frac{1+q}{1-3q}} - 1\right), \tag{1}$$

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[1] Except for the animals in Corollary 21, all objects considered in this paper are on the square lattice.

where $|A|$ denotes the number of sites in an animal, A. A similar solution exists for directed site-animals on the triangular lattice and a directed cubic lattice (in which both nearest-neighbour and next-nearest-neighbour steps are allowed).

The generating function of directed bond-animals is defined in a similar way:

$$B(z) = \sum_{\substack{A \in \ \text{directed} \\ \text{bond-animals}}} z^{|A|}, \tag{2}$$

where $|A|$ denotes the number of bonds in the animal A. Despite the similarity of the underlying objects, the directed bond animal generating function remains unsolved.

In this paper we show that a possible reason that directed bond-animals remain unsolved is that their generating function, in particular their *anisotropic* generating function, is not within the class of *differentiably finite* functions. Consequently it is fundamentally different from that of directed site-animals and most other solved bond lattice models. A similar result for self-avoiding polygons was recently given in [17]. Unfortunately this result does not then imply that the isotropic generating function is not differentiably finite.

In the next section we define differentiably finite functions and the anisotropic generating functions of directed bond and directed site-animals. In Section 3 we prove that the anisotropic generating function of directed bond-animals is not differentiably finite. An immediate corollary of this is that the generating function of directed bond-animals on the d-dimensional hypercubic lattice (with $d \geqslant 2$) is not D-finite.

2. Anisotropic and differentiably finite generating functions

Perhaps the most common functions in combinatorics and mathematical physics are those that satisfy simple linear differential equations with polynomial coefficients—these functions are called *differentiably finite* or *D-finite*. More precisely:

Definition 3. Let $f(t)$ be a formal power series in t with coefficients in $\mathbb{C}$. This series is *differentiably finite* or *D-finite* if there exist a nontrivial differential equation of the form

$$P_k(t) f^{(k)}(t) + \cdots + P_1(t) f'(t) + P_0(t) f(t) = 0, \tag{3}$$

where the $P_i(t)$ are polynomials in t with complex coefficients. It can also be shown that any algebraic power series is also a D-finite power series [20].

Ideally we would like to show that the generating function, $B(z)$, is fundamentally different in nature from that of directed site-animals, $S(q)$, which is an algebraic, and hence D-finite, power series. Perhaps the easiest way to demonstrate that a series is not D-finite is to examine its singularities; the classical theory of linear differential equations implies that D-finite series of a single variable cannot have an infinite number of singularities (see [21] for example). By this reasoning the function $f(t) = \tan(t)$ is not a D-finite power series in t.

Unfortunately, almost nothing is known rigorously about $B(z)$—we do not even know the exact location of its dominant singularity, and (the author) certainly cannot show that it has an infinite number of singularities. Fortunately, by considering the *anisotropic generating function* we are able to make considerably more progress.
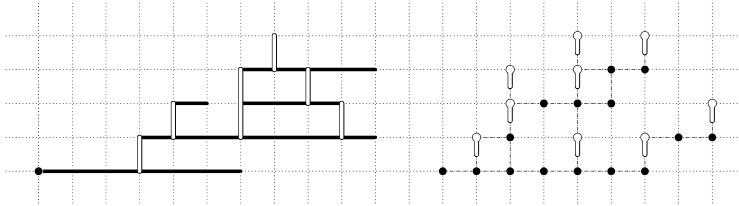
Fig. 3. (Left) A directed bond-animal with vertical bonds highlighted. (Right) A directed site-animal with sites supported only from the south highlighted.

We form the anisotropic generating function of directed bond-animals by counting animals, A, according to the number of horizontal (respectively vertical) bonds it contains, denoted $|A|_\Leftrightarrow$ (respectively $|A|_\Updownarrow$):

$$B(x, y) = \sum_{\substack{A \in \text{ directed} \\ \text{bond-animals}}} x^{|A|_\Leftrightarrow} y^{|A|_\Updownarrow} = \sum_{m,n} b_{m,n} x^m y^n, \tag{4}$$

where $b_{m,n}$ is the number of directed bond-animals with m horizontal and n vertical bonds.

Anisotropisation of the directed site-animal generating function is more problematic and is not unique. Discussions on this topic are given in [6,12]. In [6], it is suggested to anisotropise directed site-animals by counting them according to the number of sites and the number of sites *supported only from the south*—i.e. the number of sites that have a neighbour to the south, but not west. An example is given in Fig. 3. Directed site-animals on the square lattice had already been enumerated according to this parameter in [4] and results in the following algebraic generating function

$$S(q, s) = \frac{1}{2} \left(\sqrt{1 - \frac{4q}{(1+q)(1+q-qs)}} - 1 \right). \tag{5}$$

Expanding both $S(q, s)$ and $B(x, y)$ as power series in s and y (respectively) one observes a marked difference in the structure of their coefficients which leads to our main result. Rewriting $S(q, s) = \sum_{n \geq 0} R_n(q) s^n$, we find that the $R_n(q)$ are rational functions of q, and the first few are:

$$R_0(q) = \frac{q}{1-q},$$

$$R_1(q) = \frac{q^2}{(1-q)^3},$$

$$R_2(q) = \frac{q^3(1+q+q^2)}{(1-q)^5(1+q)},$$

$$R_3(q) = \frac{q^4(1+2q+4q^2+2q^3+q^4)}{(1-q)^7(1+q)^2}. \tag{6}$$

Expansion shows that the numerators are symmetric, positive and unimodal and that the denominators are given by $D_n(q) = (1-q)^{2n+1}(1+q)^{n-1}$. Hence the $R_n(q)$ are only singular at $q = \pm 1$.

Similarly, the generating function $B(x, y)$ can be rewritten as $B(x, y) = \sum_n H_n(x) y^n$, where $H_n(x)$ counts the number of directed bond-animals with n vertical bonds according to the number

of horizontal bonds they contain. Using computer enumeration techniques [13] one can find[2] the first few $H_n(x)$:

$$H_0(x) = \frac{1}{1-x},$$

$$H_1(x) = \frac{1}{(1-x)^3},$$

$$H_2(x) = \frac{1 + 2x + x^2 - x^3}{(1-x)^5(1+x)},$$

$$H_3(x) = \frac{1 + 5x + 7x^2 + x^3 - 3x^4 - 2x^5 + x^6}{(1-x)^7(1+x)^2},$$

$$H_4(x) = \frac{[1, 10, 33, 53, 43, 3, -25, -20, 1, 5, 2, -1]}{(1-x)^9(1+x)^3(1+x+x^2)}, \tag{7}$$

where we have written $[a_0, a_1, \ldots, a_n]$ in place of $a_0 + a_1 y + \cdots + a_n y^n$.

We observe that the $H_n(x)$ are simple rational functions whose denominators are products of cyclotomic polynomials.[3] This structure is quite general and can be proved using the *haruspicy* techniques described in [16]:

Theorem 4. (From [16].) *If $B(x, y) = \sum_{n \geqslant 0} H_n(x) y^n$ is the anisotropic generating function of directed bond-animals, then*

- $H_n(x)$ *is a rational function,*
- *the degree of the numerator of $H_n(x)$ cannot be greater than the degree of its denominator, and*
- *the denominator of $H_n(x)$ is a product of cyclotomic polynomials.*

If we look a little further we find that the numerators become increasingly complicated, but the denominators, which we denote $D_n(x)$, retain a regular structure. Unlike those of directed site-animals, the denominators of the coefficients of the directed bond-animal generating function contain higher and higher order cyclotomic polynomials, and hence have more and more zeros:

$$D_5(x) = (1-x)^{11}(1+x)^4(1+x+x^2)^2,$$

$$D_6(x) = (1-x)^{13}(1+x)^5(1+x+x^2)^3(1+x^2),$$

$$D_7(x) = (1-x)^{15}(1+x)^6(1+x+x^2)^4(1+x^2)^2,$$

$$D_8(x) = (1-x)^{17}(1+x)^7(1+x+x^2)^5(1+x^2)^3(1+x+x^2+x^3+x^4). \tag{8}$$

This dichotomy between the denominators of solved and unsolved models is observed in many different lattice models and was suggested as the basis of a numerical test of "*solvability*" by

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[2] More precisely, the first hundred (or so) terms of the expansion of $H_n(x)$ were fitted using Padé approximants. Construction of the approximant does not require many series terms, and the other terms serve to "verify" the conjectured form. We also note that in [16] it is proved that $H_n(x)$ is rational and bounds are given for the degrees of its numerator and denominator.

[3] We remind the reader that the cyclotomic polynomials are the factors of $(1 - x^n)$, and in particular $(1 - x^n) = \prod_{k|n} \Psi_k(x)$, where $\Psi_k(x)$ is the kth cyclotomic polynomial.

Guttmann and Enting [11,12]—if one observes an increasing number of zeros in the denominators of the coefficients of the anisotropic generating function then the model is probably not solvable. One can make this notion of solvability more precise by relating it to differentiably finite functions:

Theorem 5. (From [5].) *Let* $f(x, y) = \sum_{n \geqslant 0} y^n H_n(x)$ *be a D-finite series in* y *with coefficients* $H_n(x)$ *that are rational functions of* x. *For* $n \geqslant 0$ *let* S_n *be the set of poles of* $H_n(x)$, *and let* $S = \bigcup_n S_n$. *Then* S *has only a finite number of accumulation points.*

Consequently if the set of zeros of the denominators of the anisotropic generating function has an infinite number of accumulation points then the anisotropic generating function is not D-finite. Unfortunately Theorem 4 does not give sufficiently detailed information to prove results about the set of singularities of the coefficients, $H_n(x)$. Ideally, we would like to prove the exact form of the denominator, which appears to be

$$
D_n(x) = (1 - x)^n \prod_{k=1}^{\lfloor n/2 \rfloor + 1} \Psi_k(x)^{n-2k+3}, \tag{9}
$$

however this seems to be extremely difficult.[4] Instead we prove a weaker result that is still sufficient:

Theorem 6. *The denominator of* $H_{2k-2}(x)$ *contains a factor of* $\Psi_k(x)$ *which does not cancel with the numerator, and so* H_{2k-2} *is singular at the zeros of* $\Psi_k(x)$.

This result them implies:

Corollary 7. *The singularities of the coefficients* $H_n(x)$ *in the anisotropic generating function* $B(x, y)$ *form a dense set on the unit circle* $|x| = 1$, *and so* $B(x, y)$ *is not a D-finite power series in* y.

Since the specialisation of any D-finite power series is itself D-finite (provided the specialisation is well defined—i.e. nonsingular), we are able to extend this result to directed bond-animals on any hypercubic lattice.

3. Proof of Theorem 6

The haruspicy techniques in [16] give a way of linking the "topology" (in some loose sense) of subsets of bond-animals to the structure of their generating functions—and in particular a way of determining which "topologies" cause which singularities; the next theorem makes this idea precise. We note that we do not give definitions of *dense*, *section* and *section-minimal animal* in the main body of the paper and we refer the reader to Appendix A (or to [16]).

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[4] One can probably prove that $D_n(x)$ is a factor of the product on the right-hand side of this expression using the techniques described [16]—proofs of similar results for self-avoiding polygons and general bond animals are given in [16,17].
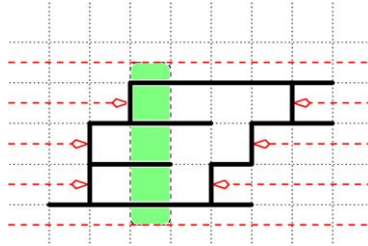
Fig. 4. A directed bond-animal with $2k - 2$ vertical bonds and a k-section (highlighted) with $k = 4$.

Theorem 8. (From [16].) *Let $\mathcal{A}_n$ be a dense set of animals with n vertical bonds. And let*

$$H_n(x) = \sum_{A \in \mathcal{A}_n} x^{|A|_\Leftrightarrow}.$$

If $H_n(x)$ has a denominator factor $\Psi_k(x)$, then there must be a section-minimal animal in $\mathcal{A}_n$ that contains a K-section for some $K \in \mathbb{Z}^+$ divisible by k. Further if $H_n(x)$ has a denominator factor $\Psi_k(x)^\alpha$, then there must be a section-minimal animal in $\mathcal{A}_n$ that contains α sections that are K-sections for some (possibly different) $K \in \mathbb{Z}^+$ divisible by k.

Note that below we shall write "*animal*" instead of "*directed bond-animal*" for convenience.

3.1. Animals that cause $\Psi_k(x)$

Theorem 6 asserts that a factor of $\Psi_k(x)$ occurs in the denominator of $H_{2k-2}(x)$. According to the above theorem this can only be the case if there is a section-minimal animal with $2k - 2$ vertical bonds that contains at least one k-section (or a K-section with K an integer multiple of k). We start by characterising such animals.

Lemma 9. *Let A be an animal that contains a k-section. A must contain at least $2k - 2$ vertical bonds. If A contains a k-section and exactly $2k - 2$ vertical bonds then there must be exactly 2 vertical bonds in each row of A.*

Proof. Consider an animal that contains a k-section. The k-section must contain at least $k - 1$ cells in a vertical line (see Fig. 4). In order to be a k-section, no section-line may cross any of these cells. Hence each section line to the left and right of these cells must be obstructed by a vertical bond and so there must be at least 1 vertical bond to the left and 1 vertical bond to the right of each of these cells. Hence an animal that contains a k-section must contain at least $2k - 2$ vertical bonds.

By similar reasoning, if the animal contains exactly $2k - 2$ vertical bonds then there must be 2 vertical bonds in each row. □

We note that one can push the above proof further to show that a section-minimal directed bond-animal with exactly $2k - 2$ vertical bonds contains no more than one k-section, however we do not need this result. We also note that the above lemma and Theorem 8 imply that the denominators of $H_n(x)$ with $n < 2k - 2$ cannot contain a factor of $\Psi_k(x)$.

The previous lemma shows that the factor of $\Psi_k(x)$ in the denominator of $H_{2k-2}(x)$ is caused by those section-minimal animals that contain a k-section, which requires that they have 2 ver-
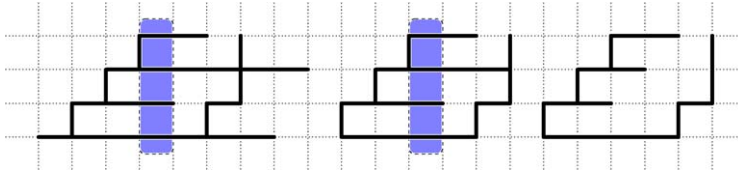
Fig. 5. (Left) A 2-directed bond-animal which has 2 vertical bonds in each row. (Centre) The corresponding *primitive* 2-directed bond animal. (Right) A 2-directed animal that contains $2k - 2$ vertical bonds but no k-section.

tical bonds in each row. In order to prove that this denominator factor does not cancel with the numerator of $H_{2k-2}(x)$, we need to examine the set of all directed bond-animals with 2 vertical bonds per row.

Definition 10. A 2-*directed animal* is a directed animal which has 2 vertical bonds in each row. A *primitive* 2-*directed animal* is a 2-directed animal in which all vertices of degree 1 lie between (or on) vertical bonds (see Fig. 5).

By Lemma 9 all animals that contain a k-section and have $2k - 2$ vertical bonds are 2-directed animals, but there are 2-directed animals with $2k - 2$ vertical bonds that do not contain a k-section (see Fig. 5). Also, one may construct a 2-directed animal from a *primitive* 2-directed animal by prepending a line of horizontal bonds to the left of the bottom-leftmost-vertex, and appending lines of horizontal bonds to the right of the rightmost vertices. Consequently, if $f_n(x)$ is the generating function of *primitive* 2-directed animals with $2n$ vertical bonds, then $(\frac{1}{1-x})^{n+2} f_n(x)$ is the generating function of all 2-directed animals with $2n$ vertical bonds.

Lemma 11. *The generating function of 2-directed animals with $2n - 2$ vertical bonds $(n > 0)$ has poles at the zeros of $\Psi_n(x)$ if and only if $H_{2n-2}(x)$ has poles at the zeros of $\Psi_n(x)$.*

Proof. Since section deletion and duplication do not alter the number of vertical bonds, nor move them between rows, it follows that 2-directed animals are closed under section duplication and deletion and so form a dense set. Similarly the set of directed bond animals that are not 2-directed bond-animals is dense. This means that we may apply Theorem 8 to both of these sets.

Let $\mathcal{A}$ be the set of all directed bond-animals with $2n - 2$ vertical bonds, and let $\mathcal{B}$ be the set of all 2-directed animals with $2n - 2$ vertical bonds. Now split $H_n(x)$ into a sum over the animals in $\mathcal{B}$ and all the others:

$$H_{2n-2}(x) = \sum_{A \in \mathcal{B}} x^{|A|_\Leftrightarrow} + \sum_{A \in \mathcal{A} \setminus \mathcal{B}} x^{|A|_\Leftrightarrow} = G_1(x) + G_2(x). \tag{10}$$

By Theorem 4 we know that $G_1(x)$ and $G_2(x)$ are rational generating functions whose denominators are products of cyclotomic polynomials. Since all those section-minimal animals with k-sections contribute to $G_1(x)$ and not $G_2(x)$, by Theorem 8 there is no factor of $\Psi_n(x)$ (or higher cyclotomic factors) in the denominator of $G_2(x)$.

Let $G_1(x)$ have a factor of $\Psi_n(x)^\alpha$ in its denominator that does not cancel with its numerator. Since there are no factors of $\Psi_n(x)$ in the denominator of $G_2(x)$, it follows that $H_{2n-2}(x)$ also has a factor of $\Psi_n(x)^\alpha$ in its denominator. Similarly if $H_n(x)$ has a factor of $\Psi_n(x)^\alpha$ in its denominator that does not cancel with its numerator, then so must $G_1(x)$. □

The above lemma makes the proof of Theorem 6 much simpler. Instead of having to analyse all directed bond-animals, we only need look at a much simpler subset—2-directed animals. Furthermore, in order to analyse this subset we do not need to explicitly enumerate it nor find an explicit expression for its generating function; it suffices to locate the singularities of its generating function.

3.2. Counting 2-directed animals

In order to study the generating function of 2-directed animals we make use of a powerful enumeration technique, the Temperley method. The method consists (essentially) of two steps—finding a recurrence satisfied by coefficients or generating functions, and then solving that recurrence. For the purposes of this paper we need to analyse the singularities of the generating function, and it transpires that an expression for the generating function is unnecessary—it is sufficient to work with the recurrences it satisfies. As was the case in [17] we use a variation of the Temperley method involving Hadamard products.

We start by defining the restricted Hadamard product and then showing how it may be used to find a recurrence satisfied by the generating function of 2-directed animals.

Definition 12. Let $f(t) = \sum_{n \geqslant 0} f_n t^n$ and $g(t) = \sum_{n \geqslant 0} g_n t^n$ be formal power series in t. The (restricted) Hadamard product is defined to be

$$f(t) \odot_t g(t) = \sum_{n \geqslant 0} f_n g_n.$$

We note that if $f(t)$ and $g(t)$ are two power series with real coefficients such that

$$\lim_{n \to \infty} |f_n g_n|^{1/n} < 1,$$

then the Hadamard product $f(t) \odot_t g(t)$ will exist.

Below we consider Hadamard products of power series in t whose coefficients are power series in two variables x and s. The products are of the form $f(t; x) \odot_t T(t, s; x) = \sum_{n \geqslant 0} f_n(x) T_n(s; x)$. The summands are the generating functions of certain directed bond animals and it follows that the nth summand is $O(sx^n)$ and so the sum converges. In order to re-express the Hadamard products we will use the following result:

Lemma 13. *Let $f(t)$ be a formal power series in t. The following (restricted) Hadamard products are easily evaluated*:

$$f(t) \odot_t \frac{1}{1 - \alpha t} = f(\alpha),$$

$$f(t) \odot_t \frac{n! t^n}{(1 - \alpha t)^{n+1}} = \frac{\partial^n f}{\partial t^n}\bigg|_{t=\alpha}.$$

We also note that the Hadamard product is a linear operator.

Proof. See similar lemma in [17]. □

Every 2-directed animal may be constructed row by row—many other objects have been counted in this way. In this paper we use the same variation of this technique used in [17] which
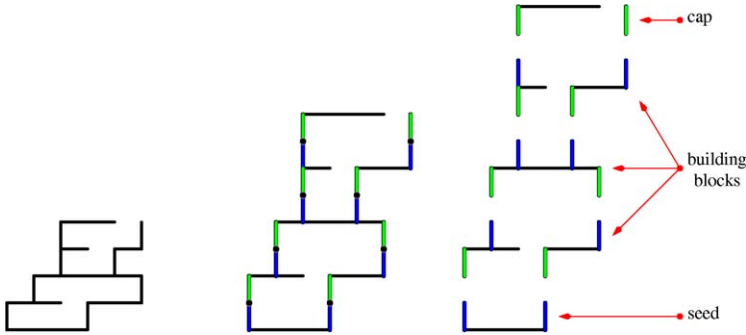
Fig. 6. Decomposing a 2-directed animal into a seed, a sequence of building blocks and then a cap.

Fig. 7. The section-minimal caps.

involves decomposing the object into a *seed* and *building blocks*. To simplify the following discussion we will work with primitive 2-directed animals rather than all 2-directed animals; since their generating functions differ only by factors of $(1-x)$, the other cyclotomic factors are unaffected. For convenience we shall drop the word "*primitive*."

Start with a 2-directed animal and duplicate every row (including the vertical bonds in each row)—see Fig. 6(left and centre). Now cut horizontally through the centre of each pair of duplicated rows; this decomposes the animal into a "*seed block*" (occupying a single row at the bottom of the animal), a sequence of "*building blocks*" (each occupying two rows) and then a "*cap*" (occupying a single row at the top of the animal)—see Fig. 6(top). We note that the sequence of blocks is restricted so that the top row of one block must have the same length as the bottom row of the next block—the Hadamard product allows us to easily translate this restriction into an operation on generating functions.

We are able to find a recurrence satisfied by the generating function of 2-directed animals from the generating functions of the seeds, building blocks, and caps. In particular we must enumerate each of these objects according to the number of horizontal bonds, and the distance between the vertical bonds.

The seed is simply a line of horizontal bonds terminated on each end by a vertical bond. It has generating function $\frac{sx}{1-sx}$ (where s is conjugate to the distance between the vertical bonds).

The caps consist of two vertical bonds with some number of horizontal bonds between them. Since the animal is directed, these horizontal bonds must be attached to the left-hand vertical bond, but not necessarily the right-hand vertical bond. The section-minimal caps are given in Fig. 7, and expanding them gives the generating function:

$$\frac{t(1+x-tx)}{(1-t)(1-tx)} = -1 + \frac{1}{(1-x)(1-t)} - \frac{x}{(1-x)(1-tx)}, \tag{11}$$

where t is conjugate to the distance between the vertical bonds.

The building blocks are (reasonably) complicated and we give the section-minimal building blocks in Fig. 8. We compute the generating function of the building blocks by expanding each of the sections. We now need two extra variables: s and t are conjugate to the distances between
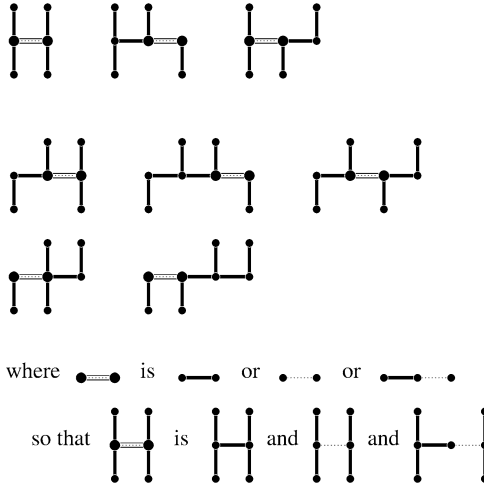
Fig. 8. The section-minimal building blocks of 2-directed bond-animals. The highlighted horizontal bonds are short-hand for either a single horizontal bond, a single horizontal space (with no bond) or a single horizontal bond and a single horizontal space.

the vertical bonds in the top and bottom rows, respectively. Doing this gives (moving from left-to-right and top-to-bottom in Fig. 8):

$$
\begin{aligned}
T(s,t;x) =\ & \big(\llbracket stx \rrbracket + \llbracket st \rrbracket + \llbracket stx \rrbracket \llbracket st \rrbracket\big) + \llbracket stx \rrbracket\big(\llbracket tx \rrbracket + \llbracket t \rrbracket + \llbracket tx \rrbracket \llbracket t \rrbracket\big) \\
& + \big(\llbracket stx \rrbracket + \llbracket st \rrbracket + \llbracket stx \rrbracket \llbracket st \rrbracket\big)\llbracket sx \rrbracket \\
& + \llbracket tx \rrbracket\big(\llbracket stx \rrbracket + \llbracket st \rrbracket + \llbracket stx \rrbracket \llbracket st \rrbracket\big) + \llbracket tx \rrbracket \llbracket stx \rrbracket\big(\llbracket tx \rrbracket + \llbracket t \rrbracket + \llbracket tx \rrbracket \llbracket t \rrbracket\big) \\
& + \llbracket tx \rrbracket\big(\llbracket stx \rrbracket + \llbracket st \rrbracket + \llbracket stx \rrbracket \llbracket st \rrbracket\big)\llbracket sx \rrbracket \\
& + \big(\llbracket tx \rrbracket + \llbracket t \rrbracket + \llbracket tx \rrbracket \llbracket t \rrbracket\big)\llbracket sx \rrbracket + \big(\llbracket tx \rrbracket + \llbracket t \rrbracket + \llbracket tx \rrbracket \llbracket t \rrbracket\big)\llbracket x \rrbracket \llbracket sx \rrbracket, \quad (12)
\end{aligned}
$$

where we have used the short hand notation $\llbracket f \rrbracket = \frac{f}{1-f}$. This may then be written in (a slightly nonstandard) partial fraction form as:

$$
\begin{aligned}
T(s,t;x) =\ & -\frac{sx}{(1-x)(1-sx)}\big(t^0\big) + \frac{2sx}{(1-x)^2(1-sx)}\left(\frac{1}{1-t}\right) \\
& + \frac{s}{(1-x)(1-sx)(s-x)}\left(\frac{1}{1-st}\right) \\
& + \frac{(s-1)s - (s-2)(s^2-s+1)sx - (s^2-s+3)s^2x^2 + (2s^2+1)sx^3 - s^2x^4}{(1-x)^2(1-s)^2(1-sx)(s-x)}\left(\frac{1}{1-tx}\right) \\
& - \frac{sx^2}{(1-x)(1-s)}\left(\frac{t}{(1-tx)^2}\right) + \frac{s(1-(1+x-x^2)s)}{(1-x)(1-s)^2(1-sx)}\left(\frac{1}{1-stx}\right), \quad (13)
\end{aligned}
$$

which we shall rewrite (more concisely) as:

$$
\begin{aligned}
T(s,t;x) =\ & c_0\big(t^0\big) + c_1\left(\frac{1}{1-t}\right) + c_2\left(\frac{1}{1-st}\right) + c_3\left(\frac{1}{1-tx}\right) \\
& + c_4\left(\frac{t}{(1-tx)^2}\right) + c_5\left(\frac{1}{1-stx}\right), \quad (14)
\end{aligned}
$$

where the c_i are the corresponding rational functions of s and x.

We find a recurrence for 2-directed animals in two steps. First we consider "*uncapped*" 2-directed animals, which are those generated from a seed and sequence of building blocks, but no cap. These are simply 2-directed animals with no horizontal bonds attached to the top of vertical bonds in their topmost row. We then find a recurrence for all 2-directed animals in terms of the uncapped 2-directed animals.

Lemma 14. *Let $\check{f}_n(s; x)$ be the generating function of* uncapped 2-*directed bond-animals* (*with* $2n$ *vertical bonds*). *The variable x is conjugate to the number of horizontal bonds and s is conjugate to the distance between the vertical bonds in the topmost row. This generating function satisfies the following functional equation*:

$$\check{f}_1(s; x) = \frac{sx}{1 - sx}, \tag{15}$$

$$\check{f}_{n+1}(s; x) = c_1 \check{f}_n(1; x) + c_2 \check{f}_n(s; x) + c_3 \check{f}_n(x; x) + c_4 \left.\frac{\partial \check{f}_n}{\partial s}\right|_{s=x} + c_5 \check{f}_n(sx; x), \tag{16}$$

where the c_i are given in Eqs. (13) *and* (14).

This recurrence is singular at two points of interest (*for results below*), *namely $s = 1$ and $s = x$. At these points singularities of the building block generating function coalesce and the recurrences change structure*:

$$\check{f}_{n+1}(1; x) = \frac{1 + 2x}{(1 - x)^3} \check{f}_n(1; x) - \frac{1 + x + x^2}{(1 - x)^3} \check{f}_n(x; x) - \frac{x(1 + x)}{(1 - x)^2} \left.\frac{\partial \check{f}_n}{\partial s}\right|_{s=x}$$

$$- \frac{x^3}{2(1 - x)} \left.\frac{\partial^2 \check{f}_n}{\partial s^2}\right|_{s=x}, \tag{17}$$

$$\check{f}_{n+1}(x; x) = \frac{2x^2}{(1 - x)^2(1 - x^2)} \check{f}_n(1; x) - \frac{x}{(1 - x)^3} \check{f}_n(x; x) + \frac{x(1 - x^2 - x^3)}{(1 - x)(1 - x^2)} \left.\frac{\partial \check{f}_n}{\partial s}\right|_{s=x}$$

$$+ \frac{x}{(1 - x)^2} \check{f}_n(x^2; x). \tag{18}$$

We note that the recurrence is also singular when $s = 1/x$, however we do not need to study it at this value of s for the results below.

Proof. The generating function of uncapped 2-directed animals with 2 vertical bonds is exactly that of the seed generating function, namely $\frac{sx}{1-sx}$. We then obtain the generating functions, $\check{f}_n(s; x)$, by repeatedly adding building blocks.

Let $\check{f}_n(s; x) = \sum_{m \geqslant 1} \check{f}_{n,m}(x) s^m$, and $T(s, t; x) = \sum_{m \geqslant 1} T_m(s; x) t^m$. The coefficient $\check{f}_{n,m}(x)$ counts those 2-directed animals which have m cells separating the 2 vertical bonds in their *top* row. Similarly $T_m(s; x)$ counts those building blocks with m cells separating the 2 vertical bonds in their *bottom* row. Thus adding a new building block corresponds to the following operation on the generating functions:

$$\check{f}_{n+1}(s; x) = \sum_{m \geqslant 1} \check{f}_{n,m}(x) T_m(s; x) = \check{f}_n(t; x) \odot_t T(s, t; x).$$

Applying Lemma 13 to the partial fraction form of $T(s, t; x)$ gives the first recurrence. Repeating this with $s = 1$ and $s = x$ gives the later recurrences. Note that $\check{f}_n(0; x) = 0$, since there must be some positive number of cells separating the vertical bonds in the top row of the animal. □

Lemma 15. *The generating function, $f_n(x)$, of all 2-directed animals with $2n$ vertical bonds may be expressed in terms of the generating function of uncapped 2-directed animals*:

$$f_n(x) = \frac{1}{1-x}\left(\check{f}_n(1;x) - x\,\check{f}_n(x;x)\right). \tag{19}$$

Proof. By similar reasoning to that given in the proof of the previous lemma, we may express the capped generating function as a Hadamard product of the uncapped generating functions together with the generating function of the caps (see Eq. (11)). Again we make use of the fact that $\check{f}_n(0;x) = 0$. □

3.3. Analysing the singularities

Using the recurrences for the generating functions of 2-directed bond-animals we proceed in two steps. We iterate the recurrences in order to determine the structure of the coefficients as rational functions of s and x. We then substitute this structure back into the recurrence to link the singularities of the generating function of animals with $2n$ vertical bonds at $s = 1$ to those with $2n - 2$ vertical bonds at $s = x$. Continuing this reasoning, we link the singularities of f_n—a function we do not know in closed form—to the singularities of f_1—which is a simple rational function that we do know.

Examining the first few generating functions, $\check{f}_n(s;x)$ we see that their denominators are products of cyclotomic polynomials, $\Psi_k(x)$, and factors of the form $(1 - sx^k)$. To refer easily to polynomials of this type we define the following sets:

Definition 16. Let $\mathbb{D}_n(s;x)$ be the set of all polynomials of the form

$$(x - s)^{a_{-1}}(1 - s)^{a_0}\prod_{k=1}^{n-1}\left(1 - sx^k\right)^{a_k}\prod_{k=1}^{n}\Psi_k(x)^{b_k}, \tag{20}$$

where a_k and b_k are nonnegative integers. Note that the factor $(1 - sx^n)$ cannot occur in a polynomial in $\mathbb{D}_n(s;x)$.

Using the above notation we can describe the structure of these generating functions:

Lemma 17. *The generating function, $\check{f}_n(s;x)$, is of the form*:

$$\check{f}_n(s;x) = \frac{N_n(s;x)}{D_n(s;x)(1 - sx^n)}, \tag{21}$$

where $N_n(s;x)$ and $D_n(s;x)$ are polynomials in s and x. Further $D_n(s;x)$ is a polynomial in $\mathbb{D}_n(s;x)$ which does not contain factors of $(1 - s)$ or $(s - x)$.

Proof. Since $\check{f}_1(s;x) = \frac{sx}{1-sx}$, the lemma is true for $n = 1$. We proceed by induction on n. Assume that $\check{f}_n(s;x)$ is of the desired form. We prove that $\check{f}_{n+1}$ is of the desired form using the recurrences in Lemma 14.

Since the denominator of $\check{f}_n(s;x)$ does not contain factors of $(1 - s)$ or $(x - s)$ we find that $\check{f}_n(1;x)$, $\check{f}_n(x;x)$ and $\frac{\partial \check{f}_n}{\partial s}|_{s=x}$ are all of the form $\frac{N(x)}{D(x)}$ where $N(x)$ and $D(x)$ are polynomials in

x with the restriction that $D(x)$ is a product of the first $n+1$ cyclotomic polynomials. Substituting this form into Eq. (16) we see that

$$\check{f}_{n+1}(s;x) \quad \text{is of the form} \quad \frac{N_{n+1}(s;x)}{D_{n+1}(s;x)(1-sx^{n+1})},$$

with N_{n+1} and D_{n+1} being polynomials in s and x. The factor of $(1-sx^{n+1})$ in the denominator of $\check{f}_{n+1}$ comes from the $\check{f}_n(sx;x)$ term in Eq. (16).

We now show that $D_{n+1}(s;x)$ is an element of $\mathbb{D}_{n+1}(s;x)$. The denominator factors arise from two contributions—those of the coefficients, and those from the various $\check{f}_n$ terms. All of these factors lie in $\mathbb{D}_{n+1}(s;x)$ excepting the single factor of $(1-sx^{n+1})$ which comes from the $\check{f}_n(sx;x)$ term. Hence $D_{n+1}(s;x) \in \mathbb{D}_{n+1}(s;x)$.

It remains to show that $D_{n+1}(s;x)$ does not contain factors of $(1-s)$ or $(s-x)$. This is equivalent to showing that $\check{f}_{n+1}(s;x)$ is finite at $s=1$ and $s=x$. By Eq. (17) we can write $\check{f}_{n+1}(1;x)$ in terms of $\check{f}_n(1;x)$, $\check{f}_n(x;x)$, $\frac{\partial \check{f}_n}{\partial s}|_{s=x}$ and $\frac{\partial^2 \check{f}_n}{\partial s^2}|_{s=x}$ which are, by the assumed form of $\check{f}_n$, finite. Hence $\check{f}_{n+1}(1;x)$ is finite, and so the denominator of $\check{f}_{n+1}(s;x)$ does not contain a factor of $(1-s)$. A similar argument shows that $\check{f}_{n+1}(s;x)$ does not contain a factor of $(s-x)$. $\square$

Before we can substitute the above form into the recurrences satisfied by $\check{f}_n$, we need to show that one of the coefficients of the recurrence does not have zeros on the unit circle which could potentially cancel singularities of $\check{f}_n$.

Lemma 18. *At $s=x^n$, the coefficient $c_5(x^n;x) = \frac{x^n(1-(1+x-x^2)x^n)}{(1-x)(1-x^n)^2(1-x^{n+1})}$ is nonzero everywhere on the unit-circle $|x|=1$.*

Proof. When $n=1$, $c_5(x,x) = \frac{x}{(1-x)^2}$ and so is nonzero on the unit circle. Now assume $n \geqslant 2$ and consider the zeros of the numerator polynomial $(1-(1+x-x^2)x^n) = 0$. We may rewrite this as

$$x^n = \frac{1}{1+x-x^2}. \tag{22}$$

If the polynomial has a zero on the unit circle, $x = e^{i\theta}$, then it follows that $|1+x-x^2| = 1$. This then gives

$$\big(1+\cos(\theta) - \cos(2\theta)\big)^2 + \big(\sin(\theta) - \sin(2\theta)\big)^2 = 1, \tag{23}$$

which reduces to the condition $\cos^2(\theta) = 1$. Hence the only candidates for zeros are $x = \pm 1$. Inspection of the polynomial then shows that it has a single zero at $x=1$ for all $n>1$, and that it has a single zero at $x=-1$ for odd $n>1$.

Since the denominator of $c_5(x^n,x)$ contains factors of $(1-x)$ and $(1+x)$ for all $n>1$, it follows that neither $x=1$ or $x=-1$ is a zero of the function. $\square$

Theorem 19. *For all $n \geqslant 1$, the generating function $f_n(x)$ has simple poles at the zeros of $\Psi_{n+1}(x)$.*

Proof. Fix n and let ξ be a zero of $\Psi_{n+1}(x)$. We will start by showing that $\check{f}_k(x^{n-k+1};x)$ is singular at $x=\xi$ by induction on k for fixed n. We then show that this is sufficient to prove the above theorem by linking the singularities of f_n to those of $\check{f}_n$.

Setting $k = 1$ gives $\check{f}_1(x^n; x) = \frac{x^{n+1}}{1 - x^{n+1}}$ which is singular at $x = \xi$.

We now proceed by induction on k using the recurrences of Lemma 14. Assume that $\check{f}_k(x^{n-k+1}; x)$ is singular at $x = \xi$. By Lemma 17, we may write $\check{f}_n(s; x)$ as:

$$\check{f}_k(s; x) = \frac{N_k(s; x)}{D_k(s; x)(1 - sx^k)},$$

where $N_k(s; x)$ is some polynomial in s and x and $D_k(s; x) \in \mathbb{D}_k(s; x)$ and does not contain factors $(1 - s)$ or $(s - x)$. Substitute this form into the recurrence given by Eq. (16) in Lemma 14. We may now write $\check{f}_{k+1}(s; x)$ as

$$\check{f}_{k+1}(s; x) = \frac{N(s; x)}{D(s; x)} + c_5(s; x) \check{f}_k(sx; x), \tag{24}$$

where $N(s; x)$ and $D(s; x)$ are polynomials and $D(s; x) \in \mathbb{D}_{k+1}(s; x)$. Setting $s = x^{n-k}$ (for $n - k \neq 1$) gives

$$\check{f}_{k+1}(x^{n-k}; x) = \frac{N(x^{n-k}; x)}{D(x^{n-k}; x)} + c_5(x^{n-k}; x) \check{f}_k(x^{n-k+1}; x), \tag{25}$$

where $D(x^{n-k}; x)$ is some product of the first n cyclotomic polynomials—hence it cannot be zero at $x = \xi$. By Lemma 18 we know that $c_5(x^{n-k}; x)$ is not zero at $x = \xi$. Since $\check{f}_k(x^{n-k+1}; x)$ is singular at $x = \xi$ so is $\check{f}_{k+1}(x^{n-k}; x)$.

The above inductive argument works for $k = 2, \ldots, n - 2$. When $k = n - 1$ we need to substitute $s = x$. As noted in Lemma 14, the recurrence changes structure slightly and we make use of Eq. (18) instead of Eq. (16). This gives essentially the same form:

$$\check{f}_n(x; x) = \frac{N(x; x)}{D(x; x)} + \frac{x}{(1 - x)^2} \check{f}_{n-1}(x^2; x) \tag{26}$$

where $D(x; x)$ is some product of the first n cyclotomic polynomials. Since $\check{f}_{n-1}(x^2; x)$ is singular at $x = \xi$, so is $\check{f}_n(x; x)$.

By induction we have shown that $\check{f}_n(x; x)$ is singular at $x = \xi$. Further, Lemma 17 implies that the singularity is a simple pole.

Using Lemma 15 the singularities of $\check{f}_n$ are linked to those of f_n:

$$f_n(x) = \frac{1}{1 - x} \left(\check{f}_n(1; x) - x \check{f}_n(x; x) \right). \tag{27}$$

Lemma 17 then implies that $\check{f}_n(1; x)$ is not singular at $x = \xi$ and so the simple pole of $\check{f}_n(x; x)$ at $x = \xi$ implies a simple pole in $f_n(x)$ at $x = \xi$. $\quad \square$

The above theorem gives our main result:

Corollary 20. *Since $f_n(x)$ has simple poles at the zeros of $\Psi_{n+1}(x)$, the coefficient $H_{2n}(x)$ has simple poles at the zeros of $\Psi_{n+1}(x)$ and the anisotropic generating function of directed bond-animals is not differentiably finite.*

Proof. Since the generating function of primitive 2-directed animals and 2-directed animals are related by factors of $(1 - x)$, it follows from Theorem 19 and Lemma 11 that $H_{2n}(x)$ has simple poles at the zeros of $\Psi_{n+1}(x)$.

Let S be the union of the singularities of $H_n(x)$ for all n. For any $q \in \mathbb{Q}$ there exists k such that $\Psi_k(e^{2\pi i q}) = 0$, and since $H_{2k-2}(x)$ has simple poles at the zeros of $\Psi_k(x)$, it follows that

$e^{2\pi iq} \in S$. Consequently S is dense on the unit circle $|x| = 1$, and by Theorem 5 the anisotropic generating function of directed bond-animals is not differentiably finite. □

This can then be extended to give the following result:

Corollary 21. *Let $\mathcal{B}_d$ be the set of directed bond-animals on the d-dimensional hypercubic lattice, and let B_d be the anisotropic generating function*

$$B_d(x_1, \ldots, x_{d-1}, y) = \sum_{A \in \mathcal{B}_d} y^{|A|_d} \prod_{i=1}^{d-1} x_i^{|A|_i},$$

where $|A|_i$ is the number of bonds parallel to the unit vector $\vec{e}_i$. Then $B_1(y) = \frac{1}{1-y}$, and for all $d \geqslant 2$ the generating function is not a D-finite power series in y.

Proof. When $d = 1$ the only directed bond-animals consist of a line of bonds; the generating function is simply $\frac{1}{1-y}$. When $d = 2$ the result follows from the previous corollary. Finally if $d > 2$, set $x_2 = \cdots = x_{d-1} = 0$ in B_d. This specialisation is well defined since the generating function now counts those animals that are confined to the plane spanned by $\{\vec{e}_1, \vec{e}_d\}$ which are simply directed bond-animals on the square lattice.

Since the well-defined specialisation of a D-finite power series is itself D-finite [14], it follows that if B_d were a D-finite function of y, then $B(x, y)$ would also be D-finite. This contradicts the previous corollary and the result follows. □

4. Conclusion

We have demonstrated that the anisotropic generating function of directed bond-animals is not differentiably finite and so is fundamentally different from that of directed site-animals which has been solved.

Unfortunately this result does not enable us to say anything rigorous about the nature of the isotropic generating function; one can readily construct an example of a function, $f(x, y)$ which is not D-finite that becomes D-finite when $x = y$. For example:

$$F(x, y) = \sum_{n \geqslant 1} \frac{y^n}{(1 - x^n)(1 - x^{n+1})} \tag{28}$$

is not a D-finite function of y by Theorem 5. Setting $y = x = z$ gives a rational, and hence D-finite, function of z:

$$F(z, z) = \sum_{n \geqslant 1} \frac{z^n}{(1 - z^n)(1 - z^{n+1})} = \frac{z}{(1 - z)^2}. \tag{29}$$

On the other hand, the "*anisotropisation*" of models that have been solved does not alter the nature of their generating functions, rather it moves singularities around in the complex plane. Of course, this does not imply anything about unsolved problems. We note that if the isotropic generating function is indeed not D-finite then it will not be found using computer packages such as GFUN [1,19] or differential approximants [10] which can only find D-finite solutions. At best one might hope that the solution may satisfy some sort of q-linear equation.

We are currently working on extending non-D-finiteness results to other bond-animal problems including square lattice bond-animals and bond-trees. Unfortunately, work on a similar

result for self-avoiding walks appears to be beyond the scope of these techniques [18]—the self-avoiding walk analogue of 2-directed animals and 2-4-2 polygons (see [16]) appear to be quite complicated and so finding recurrences such as those in Lemma 14 would be very difficult.

Finally, it may also be possible to extend the haruspicy techniques to site-animals and poly-nominoes making it possible to show that self-avoiding polygons or general site-animals, counted by an "*anisotropised*" area are not D-finite. This would also possibly explain why directed site-animals on the hexagonal lattice remain unsolved—there is strong numerical evidence [6] indicating that their anisotropic generating function is not D-finite, and it may be possible to sharpen this evidence into proof.

Acknowledgments

Appendix A. Haruspicy

In a previous paper [16] haruspicy[5] techniques have been developed that allow us to determine properties of the anisotropic generating function of a set of bond animals without detailed knowledge of those animals. This allows the techniques to applied to problems that are unsolved, such as self-avoiding polygons [17] and (in this paper) directed bond-animals.

The basic idea is to reduce or squash the set of animals down onto some minimal set, and then determine properties of the coefficients, $H_n(x)$ of the anisotropic generating function by examining the bond configurations of the minimal animals.

We start by showing how directed bond animals may be cut up so that they may be "squashed" in a consistent way.

Definition A.1. Draw horizontal lines from the extreme left and the extreme right of the lattice towards the animal so that the lines run through the middle of each lattice cell. These lines are called *section lines*. The lines are terminated when they first touch (i.e. are obstructed by) a vertical bond (see Fig. 9).

Cut the lattice along each section line from infinity until it terminates at a vertical bond. Then from this vertical bond cut vertically in both directions until another section line is reached. In this way the animal (and the lattice) is split into *pages* (see Fig. 9); we consider the vertical bonds along these vertical cuts to lie *between* pages, while the other vertical bonds lie *within* the pages.

We call a *section* the set of horizontal bonds within a single column of a given page. Equivalently, it is the set of horizontal bonds of a column of an animal between two neighbouring section lines. A section with k horizontal bonds is a k-section. The number of k-sections in an animal, P, is denoted by $\sigma_k(P)$.

By dividing an animal into sections we see that many of the sections are superfluous and are not needed to encode its "*shape*" (in some loose sense of the word). In particular, if there are two identical sections next to each other, then we can reduce the animal by removing one of them.

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[5] The word "haruspicy" refers to techniques of divination based on the examination of the forms and shapes of the organs of animals.
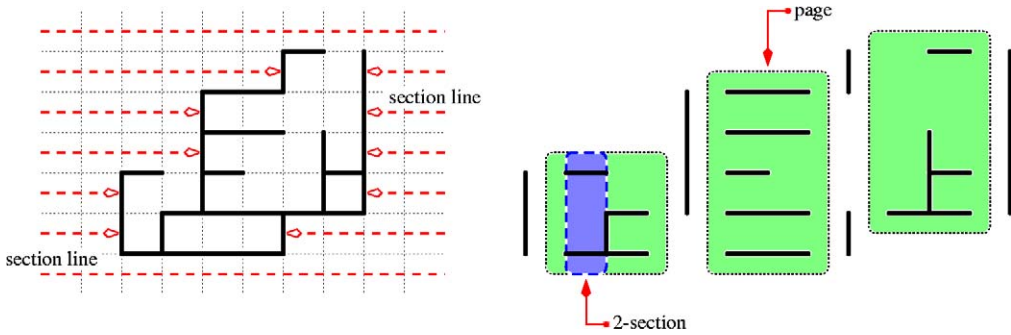
Fig. 9. *Section lines* (the heavy dashed lines in the left-hand figure) split the animal into *pages* (as shown on the right-hand figure). Each column in a page is a *section*. This animal is split into 3 pages, each containing two sections; a 2-section is highlighted. 11 vertical bonds lie between pages and 3 vertical bonds lie within the pages.
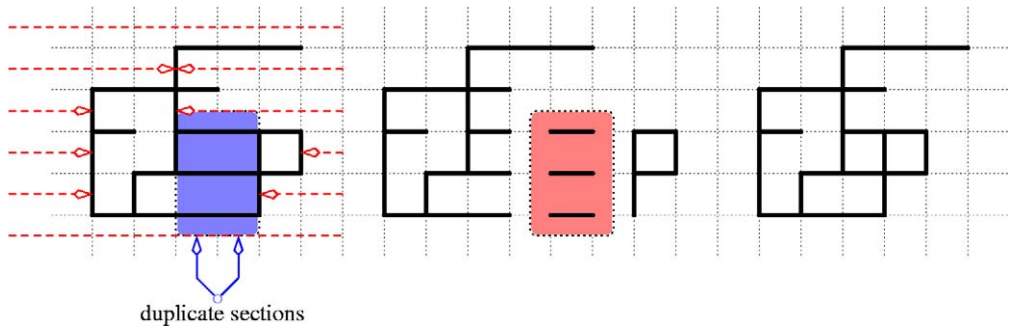

Fig. 10. The process of section deletion. The two indicated sections are identical. Slice either side of the duplicate and separate the animal into three pieces. The middle piece, being the duplicate, is removed and the remainder of the animal is recombined. Reversing the steps leads to section duplication.

Definition A.2. We say that a section is a *duplicate section* if the section immediately on its left is identical and there are no vertical bonds between them (see Fig. 10).

An animal can be reduced by *deleting* duplicate sections; slice the animal on either side of the duplicate section, remove it and then recombine it (see Fig. 10). By reversing the section-deletion process we define *duplication* of a section.

We say that a set of animals, $\mathcal{A}$, is *dense* if the set is closed under section deletion and duplication. I.e. no animal outside the set can be produced by section deletion and/or duplication from an animal inside the set.

The process of section deletion and duplication leads to a partial order on the set of animals.

Definition A.3. For any two animals P, Q, we write $P \preccurlyeq_s Q$ if $P = Q$ or P can be obtained from Q by a sequence of section deletions. A *section-minimal* animal, P, is a animal such that for all animals Q with $Q \preccurlyeq_s P$ we have $P = Q$.

The above definition leads quite directly to the following lemma:

Lemma A.4. *The binary relation $\preccurlyeq_s$ is a partial order on the set of animals. Further every animal reduces to a unique section-minimal animal, and there are only a finite number of minimal animals with n vertical bonds.*

By considering the generating function of all animals that are equivalent (by some sequence of section deletions) to a given section-minimal animal, we find that $H_n(x)$ may be written as the sum of simple rational functions. Theorem 4 follows directly from this. Further examination of the denominators of these functions gives Theorem 8. Details are given in [16].

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