

Two Non-holonomic Lattice Walks in the Quarter Plane

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Abstract

We present two classes of random walks restricted to the quarter plane whose generating function is not holonomic. The non-holonomy is established using the iterated kernel method, a recent variant of the kernel method. This adds evidence to a recent conjecture on combinatorial properties of walks with holonomic generating functions. The method also yields an asymptotic expression for the number of walks of length n .

keywords: random walks, enumeration, holonomic, generating functions, kernel method

1 Introduction

Previous studies of random walks on regular lattices had led some to conjecture that random walks in the quarter plane should all have holonomic generating functions. This is not unreasonable, given that walks confined to the half plane have algebraic generating functions ([1] proves this for directed paths, but their results extend to random walks). This was, however, disproved by Bousquet-Mélou and Petkovšec with their proof that knight's walks confined to the quarter plane are not holonomic [4].

In this paper we give two new examples of random walks whose generating functions are not holonomic. We use a technique developed in a recent study of self-avoiding walks in wedges [11] called the iterated kernel method. This is an adaptation of the kernel method [2, 10], in which we express a generating function in terms of iterates of a kernel solution. We can show that this generating function has an infinite number of poles, which is incompatible with holonomy.

The aim of this work is two-fold. The first goal is to illustrate a potentially general technique to prove non-holonomy. Secondly, this work was completed in the context of a general generating function classification of all nearest neighbour lattice walks [9], and lends evidence to a general conjecture on the connection between different symmetries of the step sets and the analytic nature of their generating functions.

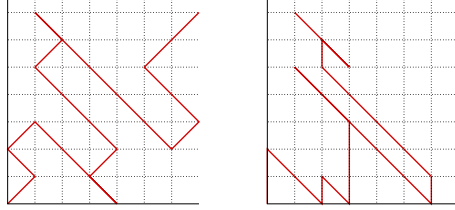


Figure 1: Sample walks with steps from $\mathcal{S} = \{\text{NW}, \text{NE}, \text{SE}\}$ and $\mathcal{T} = \{\text{NW}, \text{N}, \text{SE}\}$

1.1 Walks and their generating functions

The objects under consideration are walks on the integer lattice restricted to the quarter plane with steps taken from $\mathcal{S} = \{(-1, 1), (1, 1), (1, -1)\}$ in the first case, and $\mathcal{T} = \{(-1, 1), (0, 1), (1, -1)\}$ in the second case. We use the compass notation and label the directions of these two sets as $\mathcal{S} = \{\text{NW}, \text{NE}, \text{SE}\}$ and $\mathcal{T} = \{\text{NW}, \text{N}, \text{SE}\}$. Two sample walks are given in Figure 1.1.

We associate to these two steps sets two power series each: $W(t)$ a counting (ordinary, univariate) generating function and $Q(x, y; t)$ a complete (multivariate) generating function. The series $W(t)$, is the ordinary generating function for the number of walks. It is a formal power series where the coefficient of t^n is the number of walks of length n . The complete generating function $Q(x, y; t)$ encodes more information. The coefficient of $x^i y^j t^n$ in $Q(x, y; t)$ is the number of walks of length n ending at the point (i, j) . Remark that the specialization $x = y = 1$ in the complete generating function is precisely the counting series, i.e. $Q(1, 1; t) = W(t)$. If the choice of step set is not clear, we add an index.

In part, our interest in the complete generating function stems from the fact it satisfies a very useful functional equation which we derive using the recursive definition of a walk: a walk of length n is a walk of length $n - 1$ plus a step. The quarter plane condition asserts itself by restricting our choice of step should the smaller walk end on an axis.

The step set \mathcal{S} leads to the following equation:

$$Q(x, y; t) = 1 + t \left(xy + \frac{x}{y} + \frac{y}{x} \right) Q(x, y; t) - t \frac{x}{y} Q(x, 0; t) - t \frac{y}{x} Q(0, y; t), \quad (1.1)$$

and set \mathcal{T} defines a similar equation:

$$Q(x, y; t) = 1 + t \left(y + \frac{x}{y} + \frac{y}{x} \right) Q(x, y; t) - t \frac{x}{y} Q(x, 0; t) - t \frac{y}{x} Q(0, y; t). \quad (1.2)$$

1.2 Properties of holonomic functions

We are interested in understanding the analytic nature of the generating functions. This nature gives a basic first classification of structures, and also some general properties such as general form of the asymptotic growth of the coefficients. See for example Bousquet-Mélou's recent work on classifying combinatorial families with rational and algebraic generating functions [3]. We are interested in generating functions which are holonomic, also known as D-finite. Let $\underline{x} = x_1, x_2, \dots, x_n$.

Definition 1.1 *holonomic function.* A multivariate function $G(\underline{x})$ is *holonomic* if the vector space generated by the partial derivatives of G (and their iterates), over rational functions of \underline{x} is finite dimensional. This is equivalent to the existence of n partial differential equations of the form

$$p_{0,i}f(\underline{x}) + p_{1,i}\frac{\partial f(\underline{x})}{\partial x_i} + \dots + p_{d_i,i}\frac{\partial^{d_i} f(\underline{x})}{(\partial x_i)^{d_i}} = 0,$$

for i satisfying $1 \leq i \leq n$, and where the $p_{j,i}$ are all polynomials in \underline{x} .

The basic feature of these functions that we use is the fact that they have a finite number of singularities. For example, in the univariate case, these are poles which appear are zeroes of $p_{d_1,1}$.

1.3 Main results

We show that both the complete and counting generating functions of the walks with steps from $\mathcal{S} = \{\text{NW}, \text{NE}, \text{SE}\}$ and $\mathcal{T} = \{\text{NW}, \text{N}, \text{SE}\}$ are not holonomic using this singularity property of holonomic functions. In fact, it suffices to show that their counting generating function $W(t)$ is not holonomic, as holonomic functions are closed under algebraic substitution [8, 13], and we have $W(t) = Q(1, 1; t)$. We show that $W(t)$ has an infinite number of poles and thus is not holonomic [13]. This will give the first main theorem that we present.

Theorem 1.1. *Neither the complete generating function $Q_{\mathcal{S}}(x, y; t)$, nor the counting generating function $W_{\mathcal{S}}(t) = Q_{\mathcal{S}}(1, 1; t)$ of nearest-neighbour walks in the first quadrant with steps from $\{\text{NE}, \text{SE}, \text{NW}\}$ are holonomic functions with respect to their variable sets.*

To prove this, we use the *iterated kernel method*, as described by Janse van Rensburg *et al.* in [11] in counting partially directed walks confined to a wedge.

The basic idea is similar to other variants of the kernel method in that we write our fundamental equation in the kernel format, determine particular values of x and y which fix the kernel, and then generate new relations. For these walks, the kernel iterates form an infinite group whose elements each introduce new poles into the generating function. The main difficulty in proving that these generating functions are not holonomic is proving that the singularities do not cancel.

As step set \mathcal{T} is not symmetric in the line $x = y$, the relations obtained are more complex and while the argument is essentially the same, the details of the proof of its non-holonomy is more complicated, as we see in Section 3. This will give the second main theorem.

Theorem 1.2. *Neither the complete generating function $Q_{\mathcal{T}}(x, y; t)$, nor the counting generating function $W_{\mathcal{T}}(t) = Q_{\mathcal{T}}(1, 1; t)$ of nearest-neighbour walks in the first quadrant with steps from $\{\text{NW}, \text{N}, \text{SE}\}$ are holonomic functions with respect to their variable sets.*

2 The non-holonomy of $Q_S(x, y; t)$

We use the iterated kernel method to find a closed form expression for the generating function $Q_S(x, y; t)$. We then analyse this expression to prove this function, and the counting generating function $W_S(t)$ are not holonomic.

2.1 Defining the iterates of the kernel solutions

To begin, consider the kernel version of Equation (1.1):

$$(xy - tx^2y^2 - tx^2 - ty^2) Q(x, y) = xy - tx^2Q(x, 0) - ty^2Q(y, 0). \quad (2.1)$$

Here, for brevity we write $Q(x, y; t)$ as $Q(x, y)$, and have used the $x \leftrightarrow y$ symmetry to rewrite $Q(0, y)$ as $Q(y, 0)$.

Since the kernel $K(x, y) = xy - tx^2y^2 - tx^2 - ty^2$ is a quadratic, it has two solutions as a function of y

$$Y_{\pm 1}(x) = \frac{x}{2t(1+x^2)} \left(1 \mp \sqrt{1 - 4t^2(1+x^2)} \right). \quad (2.2)$$

Note that as formal power series in t , these roots are:

$$\begin{aligned} Y_{+1}(x) &= xt + O(t^3) \\ Y_{-1}(x) &= \frac{x}{(1+x^2)t} - xt + O(t^3). \end{aligned}$$

Further, $Y_{+1}(x)$ is a power series in t with polynomial coefficients in x . We will need to form compositions of these roots. Write $Y_n(x) = (Y_1 \circ)^n(x)$, and likewise for Y_{-n} .

Lemma 2.1. *The kernel roots obey the relation:*

$$Y_{+1}(Y_{-1}(x)) = x \quad \text{and} \quad Y_{-1}(Y_{+1}(x)) = x. \quad (2.3)$$

and the set $\{Y_n | n \in \mathbb{Z}\}$ forms a group, under the operation $Y_n(Y_m(x)) = Y_n \circ Y_m = Y_{n+m}$, with identity $Y_0 = x$. Furthermore, they also obey the relation:

$$\frac{1}{Y_1(x)} + \frac{1}{Y_{-1}(x)} = \frac{1}{tx}, \quad (2.4)$$

which extends (by substituting $x = Y_{n-1}(x)$) to:

$$\frac{1}{Y_n(x)} = \frac{1}{tY_{n-1}} - \frac{1}{Y_{n-2}(x)}. \quad (2.5)$$

Proof. To prove Equation (2.3), consider the four compositions of the kernel roots $Y_{\pm}(Y_{\pm}(x))$ and denote them by $\{C_i\}$. We form the product $\prod_{i=1}^4 (C - C_i)$, for formal parameter C , and one may verify:

$$(x - C)^2 ((t^2 + x^2)C - x(1 - 2t^2)C + t^2x^2) = 0.$$

Hence two of the compositions are the identity.

There are $\binom{4}{2}$ possibilities, and we can exclude 4 of these by consistency arguments — *i.e.* they will imply that $Y_+ = Y_-$ which is not true. This leaves us to check that either $Y_+ \circ Y_- = Y_- \circ Y_+ = x$ or $Y_+ \circ Y_+ = Y_- \circ Y_- = x$. A manual calculation then shows that $Y_- \circ Y_+ = Y_+ \circ Y_- = x$.

Equation (2.4) follows since $Y_{\pm 1}$ are roots of a quadratic — in particular $Y_{+1}Y_{-1}$ and $Y_{+1} + Y_{-1}$ are rational functions of x and t . \square

2.2 An expression for $Q(x, 0)$ in terms of the iterates

We are now able to find a formal expression for the generating function.

Theorem 2.2. *The generating function $Q(x, 0)$ is given by*

$$Q(x, 0) = \frac{1}{x^2t} \sum_{n \geq 0} (-1)^n Y_n(x) Y_{n+1}(x). \quad (2.6)$$

and so $Q(1, 1)$ is given by

$$Q(1, 1) = \frac{1 - 2tQ(1, 0)}{1 - 3t} = \frac{1 - 2 \sum_{n \geq 0} (-1)^n Y_n(1) Y_{n+1}(1)}{1 - 3t}. \quad (2.7)$$

Proof. By construction we have $K(x, Y_1(x)) = 0$, thus substituting $y = Y_1(x)$ into Eq. (2.1) gives (after a little tidying)

$$Q(x, 0) = \frac{Y_1(x)}{x} \frac{1}{t} - \frac{Y_1(x)^2}{x^2} Q(Y_1(x), 0).$$

Now we substitute $x = Y_n(x)$ into this equation to obtain

$$Q(Y_n(x), 0) = \left(\frac{Y_{n+1}(x)}{Y_n(x)} \right) \frac{1}{t} - \left(\frac{Y_{n+1}(x)}{Y_n(x)} \right)^2 Q(Y_{n+1}(x), 0).$$

Using this expression for $Q(Y_n(x), 0)$ for increasing n , we can iteratively generate a new

expression for $Q(x, 0)$:

$$\begin{aligned}
Q(x, 0) &= \frac{Y_1(x)}{x} \frac{1}{t} - \frac{Y_1(x)^2}{x^2} \left(\frac{Y_2(x)}{Y_1(x)} \frac{1}{t} - \frac{Y_2(x)^2}{Y_1(x)^2} Q(Y_2(x), 0) \right) \\
&= \frac{Y_0(x)Y_1(x)}{x^2t} - \frac{Y_1(x)Y_2(x)}{x^2t} + \left(\frac{Y_2(x)}{x} \right)^2 Q(Y_2(x), 0) \\
&= \dots \\
&= \frac{1}{x^2t} \sum_{n=0}^{N-1} (-1)^n Y_n(x) Y_{n+1}(x) + (-1)^N \left(\frac{Y_N(x)}{x} \right)^2 Q(Y_N(x), 0).
\end{aligned}$$

Since $Y_1(x) = xt + O(t^3)$ is a power series in t with positive polynomial coefficients in x , it follows that $Y_n(x) = xt^n + o(xt^n)$. Hence we have that $\lim_{N \rightarrow \infty} Y_N(x) = 0$ as a formal power series in t and the theorem follows. \square

2.3 The abundant singularities of $W(t)$

Our path to proving Theorem 1.1 is now clear. We demonstrate an infinite set of singularities for $W(t) = Q(1, 1)$ coming from the set of non-conflicting poles of $Y_n(1)$. Given equation (2.7), there are 3 potential sources of singularities for $Q(1, 1)$:

1. the simple pole at $t = 1/3$;
2. singularities from the $Y_n(1; t)$;
3. divergence of the sum.

We now show that the singularity at $t = 1/3$ is the dominant singularity and that there are an infinite number of other singularities (given by (2)), and that the series is convergent elsewhere.

Lemma 2.3. *The generating function $W_S(t)$ has a simple pole at $t = 1/3$.*

Proof. To show that $t = 1/3$ is a simple pole, we show that the numerator of equation (2.7), $1 - 2tQ(1, 0)$, is non-zero and absolutely convergent at this point.

We show in the proof of Corollary 2.7 that the radius of convergence of $Q(1, 0)$ is bounded below by $8^{-1/2} \approx 0.335$, and thus at $|t| = 1/3$ the numerator converges absolutely.

Next we show that the singularity is not removable by proving that $1 - 2tQ(1, 0)$ is non-zero at $t = 1/3$. Note that $Y_0(1; 1/3) = 1$ and $Y_1(1; 1/3) = 1/2$. Using the recurrence (2.5) for Y_n we have

$$\frac{1}{Y_n(1; 1/3)} = \frac{3}{Y_{n-1}(1; 1/3)} - \frac{1}{Y_{n-2}(1; 1/3)}$$

which leads to $Y_n(1; 1/3) = 1/F_{2n}$, i.e. the reciprocal of the $2n^{\text{th}}$ Fibonacci number. Hence $Y_n Y_{n+1} \rightarrow 0$ and so the alternating sum $Q(1, 1; 1/3)$ is convergent. Further, we may write

$$\begin{aligned} \sum_{n \geq 0} \frac{(-1)^n}{F_{2n} F_{2n+2}} &= \frac{1}{2} - \sum_{k \geq 1} \frac{F_{4k+2} - F_{4k-2}}{F_{4k-2} F_{4k} F_{4k+2}} \\ &= \frac{1}{2} - \frac{1}{10} + \sum_{k \geq 1} \frac{F_{4k+4} - F_{4k}}{F_{4k} F_{4k+2} F_{4k+4}}. \end{aligned} \quad (2.8)$$

Hence the sum is bounded strictly between $2/5$ and $1/2$ and $1 - 2Q(1, 0; 1/3)/3$ is non-zero, and thus $t = 1/3$ is a simple pole of $Q(1, 1)$. \square

2.4 The singularities are distinct

Next, we show that this function possesses an infinite number of singularities — coming from simple poles of the Y_n . In order to do this we make the substitution $t \mapsto \frac{q}{1+q^2}$ which allows us to write Y_n in closed form.

Lemma 2.4. Define $\bar{Y}_n(q) := \left(Y_n \left(1; \frac{q}{1+q^2} \right) \right)^{-1}$. Then

$$\begin{aligned} \bar{Y}_n(q) &= \left(\frac{q - \bar{Y}_1(q)}{1 - q^2} \right) \cdot q^{1-n} + \left(\frac{1 - q\bar{Y}_1(q)}{1 - q^2} \right) \cdot q^n \\ &= \left(\frac{1 - q^{2n}}{1 - q^2} \right) \cdot q^{1-n} \cdot \bar{Y}_1(q) - \left(\frac{1 - q^{2n-2}}{1 - q^2} \right) \cdot q^{2-n}. \end{aligned} \quad (2.9)$$

Proof. Substitute $t = \frac{q}{1+q^2}$ into equation (2.5). The roots of the characteristic equation are simply q and $1/q$ and the recurrence can be solved using standard methods. \square

Since we now have Y_n in closed form we can determine some facts about the location of its poles.

Lemma 2.5. Suppose q_c is a zero of $\bar{Y}_n(q) := Y_n \left(1; \frac{q}{1+q^2} \right)^{-1}$, and that $q_c \neq 0$. Then

1. q_c is a solution of $(q^{2n} + q^{-2n}) + (q^2 + q^{-2}) = 4$;
2. q_c is not a k^{th} root of unity for any k ; and
3. for all $k \neq n$, $\bar{Y}_k(q_c) \neq 0$.

Proof. If $\bar{Y}_n(q_c) = 0$ we must have

$$q_c(1 - q_c^{2n})\bar{Y}_1(q_c) = (q_c^2 - q_c^{2n}).$$

Using the equation $\bar{Y}_1(q) = \left(1 + q^2 + \sqrt{1 - 6q^2 + q^4}\right) / 2q$, this can be rearranged to give

$$\begin{aligned} (1 - q_c^{2n}) \left(1 + q_c^2 + \sqrt{1 - 6q_c^2 + q_c^4}\right) &= 2(q_c^2 - q_c^{2n}), \\ (1 - q_c^{2n}) \sqrt{1 - 6q_c^2 + q_c^4} &= 2(q_c^2 - q_c^{2n}) - (1 - q_c^{2n})(1 + q_c^2) \\ &= -(1 - q_c^2)(1 + q_c^{2n}), \\ \text{and so, } (1 - q_c^{2n})^2(1 - 6q_c^2 + q_c^4) &= (1 - q_c^2)^2(1 + q_c^{2n})^2. \end{aligned}$$

We expand and collect terms to arrive at the equation

$$q_c^{4n} + q_c^{2n} \left(\frac{1 - 4q_c^2 + q_c^4}{q_c^2}\right) + 1 = 0,$$

which can in turn be reduced to

$$(q_c^{2n} + q_c^{-2n}) + (q_c^2 + q_c^{-2}) = 4.$$

It is clear that $q = \pm 1$ are solutions of the above equation. However, substituting $q = \pm 1$ into equation (2.9) then we find $Y_1(\pm 1) = \pm(1 - i)/2$ and

$$\begin{aligned} \bar{Y}_n(1) &= n\bar{Y}_1(1) - (n - 1) \\ \bar{Y}_n(-1) &= (-1)^{n+1} (n\bar{Y}_1(-1) - (n - 1)). \end{aligned} \tag{2.11}$$

Hence neither equation can be satisfied and $q = \pm 1$ are not a poles of Y_n . Now if $q_c = e^{i\theta}$, it follows that $\cos(2n\theta) + \cos(2\theta) = 2$ so that the only possible zeros on the unit circle are at $q_c = \pm 1$, which we have already excluded. All the zeros lie off the unit circle.

Next, we prove part (3) of the lemma. First consider the possibility that $\bar{Y}_{n+1}(q_c) = 0$. If this is true, since the sequence of \bar{Y}_n satisfies a homogeneous three-term recurrence, it follows that $\bar{Y}_{n+k}(q_c) = 0$ for all $k \geq 0$. It also follows that all $\bar{Y}_{n-k}(q_c)$ must be zero. However, we can compute the zeros of $\bar{Y}_1(q)$ explicitly and the only possible zero is at $q = 0$. Hence either $q_c = 0$ or $\bar{Y}_{n+1}(q_c) \neq 0$.

Using the recurrence $\bar{Y}_k(q_c) = \left(\frac{1+q_c^2}{q_c}\right) \bar{Y}_{k-1}(q_c) - \bar{Y}_{k-2}(q_c)$, and the hypothesis that $\bar{Y}_n(q_c) = 0$, we solve the recurrence for the $\bar{Y}_{n+k}(q_c)$. This gives

$$\bar{Y}_{n+k}(q_c) = \bar{Y}_{n+1}(q_c) \frac{1 - q_c^{2k}}{(1 - q_c^2)q_c^{k-1}}. \tag{2.12}$$

Since $\bar{Y}_{n+1}(q_c) \neq 0$ by our earlier remark, and since q_c is not a root of unity, we have that $\bar{Y}_{n+k}(q_c) \neq 0$.

By iterating backwards one can similarly show that $\bar{Y}_{n-k}(q_c) \neq 0$. In fact, one can show that $\bar{Y}_{n+k}(q_c) + \bar{Y}_{n-k}(q_c) = 0$. \square

Next we show that any such zero is a pole of $Q\left(1, 0; \frac{q}{1+q^2}\right)$, and hence of $Q\left(1, 1; \frac{q}{1+q^2}\right)$

Lemma 2.6. *Let $q = q_c \neq 0$ be a zero of \overline{Y}_n . The function $Q\left(1, 0; \frac{q}{1+q^2}\right)$ has a pole at the $q = q_c$.*

Proof. First, recall that as $Y_k = q^k + O(q^{k+1})$, this is a convergent expression (as a formal power series), and thus it makes sense to consider this series in this arrangement. We contend that the only poles come from the poles given by zeroes of \overline{Y}_n , for all n . Aside from these poles and the square root singularity given by $q = \pm \frac{1}{\sqrt{8}}$, the series is convergent.

Write $Y_k Y_{k+1}$ as

$$Y_k Y_{k+1} = (1 - q^2)(1 - Y_1 q) \left(\frac{1}{q^{2n}(Y_1 - q) - Y_1 + q} - \frac{1}{q^{2n+1}(qY_1 - 1) - Y_1 + q} \right). \quad (2.13)$$

Suppose q is such that the denominator is not zero, and such that it does not lie on the branch cut of Y_1 .

We know that $|q_c| \neq 1$, so we only need to consider $|q| > 1$ and $|q| < 1$. The above expression for $Y_k Y_{k+1}$ is obtained by substituting $t \mapsto \frac{q}{1+q^2}$ and so is necessarily invariant under $q \mapsto 1/q$. Hence we only need to consider $|q| > 1$. In this case the denominators in the right-hand side of the expression are dominated by the q^k term and so approach 0 as $k \rightarrow \infty$. Since the sum is alternating it converges.

Now, the denominator is annihilated in this case precisely when q is a zero of either $\overline{Y}_k(q)$ or $\overline{Y}_{k+1}(q)$. Suppose q_c is a zero of $\overline{Y}_k(q)$. Then, by Lemma 2.5, $\overline{Y}_n(q_c) \neq 0$ if $n \neq k$. Thus, the only two terms for which q_c is singular are $Y_{k-1}Y_k$ and $Y_k Y_{k+1}$. If we sum these two terms we have $(-1)^k Y_k (Y_{k+1} - Y_{k-1})$. By our earlier comments, $Y_{k-1}(q_c) = -Y_{k+1}(q_c) \neq 0$, and thus this singularity does not cancel. \square

2.5 Proof of Theorem 1.1

We now have all the components in place to prove the main result.

Proof of Theorem 1.1. The function $Q\left(1, 1; \frac{q}{1+q^2}\right)$ has a set of poles given by the zeroes of the \overline{Y}_n , by the preceding lemma. By Lemma 2.5, this is an infinite set. Thus, $Q\left(1, 1, \frac{q}{1+q^2}\right)$ is not holonomic. For a multivariate series to be holonomic, any of its algebraic specializations must be holonomic, and as $Q\left(1, 1, \frac{q}{1+q^2}\right)$ is an algebraic specialization of both $Q(x, y)$ and $Q(1, 1)$, neither of these two functions are holonomic either. \square

Corollary 2.7. *The number of random walks of length n with step set $\{\text{NE}, \text{SE}, \text{NW}\}$ confined to the quarter plane is asymptotic to*

$$c_n \sim \alpha 3^n + O(8^{n/2}) \quad (2.14)$$

where α is a constant given by

$$\alpha = 1 - 2 \sum_{n \geq 0} \frac{(-1)^n}{F_{2n} F_{2n+2}} = 0.1731788836 \dots \quad (2.15)$$

Proof. We need to show that the dominant singularity is the simple pole at $t = 1/3$. Let us rewrite equation (2.1) at $x = y = 1$:

$$(1 - 3t)Q(1, 1) = 1 - 2tQ(1, 0) \quad \text{or} \quad Q(1, 1) = \frac{1 - 2Q(1, 0)}{1 - 3t}. \quad (2.16)$$

Hence there are 2 sources of singularities for $Q(1, 1)$ — the simple pole at $t = 1/3$ and the singularities of $Q(1, 0)$. The residue at the simple pole may be computed quite directly (see the proof of Lemma 2.3).

We now bound the singularities of $Q(1, 0)$ away from $|t| = 1/3$. The series $Q(1, 0)$ enumerates random walks confined to the quarter plane which end on the line $y = 0$. Consider a family of random walks with the same step-set but ending on the line $y = 0$ but no longer confined to the quarter plane, rather just confined to the half-plane $y \geq 0$. The generating function of these walks obeys the functional equation

$$P(y) = 1 + t \left(2y + \frac{1}{y} \right) P(y) - t \frac{1}{y} P(0). \quad (2.17)$$

The generating function of these walks can be computed using the kernel method to give:

$$P(0) = \frac{1 - \sqrt{1 - 8t^2}}{4t^2}. \quad (2.18)$$

Hence the number of these walks grows as $O(8^{n/2})$. Consequently the walks enumerated by $Q(1, 0)$ cannot grow faster than $O(8^{n/2})$, and so the radius of convergence of $Q(1, 0)$ is bounded below by $8^{-1/2}$. The result follows. \square

3 The non-holonomy of $Q_{\mathcal{T}}(x, y; t)$

This step set is not symmetric across the line $x = y$, and thus as we lose $x \leftrightarrow y$ symmetry in the complete generating function, we expect a more complicated scenario. Nonetheless, we can recycle a good number of the results from the previous case.

3.1 Defining two sets of iterates

We begin by rewriting the fundamental equation in kernel form:

$$K(x, y)Q(x, y) = xy - txQ(x, 0) - ty^2Q(0, y), \quad (3.1)$$

with kernel $K(x, y) = xy - t(xy^2 + x^2 + y^2)$. We will require zeros of the kernel for both fixed x and fixed y , which we denote respectively with X and Y . This gives,

$$X_{\pm 1}(y) = \frac{y}{2t} \left(1 \mp \sqrt{(1 - ty)^2 - t^2} \right), \quad \text{and} \quad (3.2)$$

$$Y_{\pm 1}(x) = \frac{x}{2t(1+x)} \left(1 \mp \sqrt{1 - 4t^2(x+1)} \right). \quad (3.3)$$

Using similar argument to those in Section 2, one can show that

$$X_{\pm}(Y_{\mp}(x)) = x \quad \text{and} \quad Y_{\pm}(X_{\mp}(y)) = y \quad (3.4)$$

and that

$$\frac{1}{X_{+1}(y)} + \frac{1}{X_{-1}(y)} = \frac{1}{ty} - 1 \quad (3.5)$$

$$\frac{1}{Y_{+1}(x)} + \frac{1}{Y_{-1}(x)} = \frac{1}{tx}. \quad (3.6)$$

We can use these to find nice expressions for compositions of X 's and Y 's. Define $B_n(y)$ and $A_n(x)$ by

$$\begin{aligned} B_0(y) &= y \\ B_{2k+1}(y) &= X_1(B_{2k}(y)) \\ B_{2k+2}(y) &= Y_1(B_{2k+1}(y)) \end{aligned} \quad (3.7)$$

and

$$\begin{aligned} A_0(x) &= x \\ A_{2k+1}(x) &= Y_1(A_{2k}(x)) \\ A_{2k+2}(x) &= X_1(A_{2k+1}(x)). \end{aligned} \quad (3.8)$$

As before, we can compose these relations and determine the recurrences

$$\begin{aligned} \frac{1}{B_{n+1}(y)} + \frac{1}{B_{n-1}(y)} &= \frac{1}{tB_n(y)} - 1 \\ \frac{1}{A_{n+1}(x)} + \frac{1}{A_{n-1}(x)} &= \frac{1}{tA_n(x)}. \end{aligned} \quad (3.9)$$

Note that A_k satisfies the same recurrence and initial conditions as Y_k in Lemma 2.1 of Section 2

3.2 Expressing the series in terms of the iterates

Next, we use the relations such as $K(A_{2k-1}(x), A_{2k}(x)) = 0$ and $K(A_{2k}(x), A_{2k+1}(x)) = 0$ to generate an infinite set of equations. For example, if we set $y = A_1(x) = Y(x)$, the functional equation Eq. (3.1) reduces to

$$Q(x, 0) = \frac{Y_1(x)}{xt} - \left(\frac{Y_1(x)}{x} \right)^2 Q(0, Y_1(x)),$$

and when $x = B_1(y) = X_1(y)$

$$Q(0, y) = \frac{X_1(y)}{yt} - \left(\frac{X_1(y)}{y} \right)^2 Q(X_1(y), 0).$$

Putting $y = Y_1(x)$ into the above expression for $Q(0, y)$ gives

$$Q(0, Y_1(x)) = \frac{X_1(Y_1(x))}{Y_1(x)t} - \left(\frac{X_1(Y_1(x))}{Y_1(x)} \right)^2 Q(X_1(Y_1(x)), 0).$$

Rewriting everything in terms of $A_n(x)$ and substituting one equation into the other we get

$$\begin{aligned} Q(x, 0) &= \frac{A_0(x)A_1(x)}{A_0(x)^2} \frac{1}{t} - \left(\frac{A_1(x)}{A_0(x)} \right)^2 \left(\frac{A_2(x)}{A_1(x)} \frac{1}{t} - \left(\frac{A_2(x)}{A_1(x)} \right)^2 Q(A_2(x), 0) \right) \\ &= \frac{A_0(x)A_1(x) - A_1(x)A_2(x)}{A_0(x)^2} \frac{1}{t} + \left(\frac{A_2(x)}{A_0(x)} \right)^2 Q(A_2(x), 0). \end{aligned}$$

Iterating this procedure leads to the following theorem

Theorem 3.1. *The generating function for the walks that end on either axis and the counting generating function are given by the following expressions:*

$$\begin{aligned} Q(x, 0) &= \frac{1}{x^2t} \sum_{n \geq 0} (-1)^n A_n(x) A_{n+1}(x), \\ Q(0, y) &= \frac{1}{y^2t} \sum_{n \geq 0} (-1)^n B_n(y) B_{n+1}(y), \\ Q(1, 1) &= \frac{1}{1-3t} \left(1 - \sum_{n \geq 0} (-1)^n A_n(1) A_{n+1}(1) - \sum_{n \geq 0} (-1)^n B_n(1) B_{n+1}(1) \right). \end{aligned} \tag{3.10}$$

3.3 A source of singularities

We now show that the generating function has an infinite number of poles, and that these are given by the A_n , and the B_n . We consider the series under the same q -transformation that we used when we considered the other step set. We set $\bar{A}_n(q) = A_n(1; \frac{q}{1+q^2})^{-1}$ and $\bar{B}_n(q) = B_n(1; \frac{q}{1+q^2})^{-1}$.

Lemma 3.2. *The function $\bar{A}_n(q)$ is identical to $\bar{Y}_n(q)$.*

Proof. When $x = 1$, the kernels for the functional equations for both step sets are the same. This implies that when $x = 1$, the solutions to the respective kernels are the same; That is, Y_+ is the same for both step sets and Y_- is the same for both step sets.

Consider the recurrence satisfied by the \bar{A}_n . Iterating Eq. (3.9) gives

$$\bar{A}_n(q) = \left(\frac{1 - q^{2n}}{1 - q^2} \right) \frac{1}{q^{n-1}} \bar{A}_1(q) - \left(\frac{1 - q^{2n-2}}{1 - q^2} \right) \frac{1}{q^{n-2}}, \tag{3.11}$$

and thus, $\bar{A}_n(q)$ here, satisfies the same recurrence as $\bar{Y}_n(q)$ given in Eq. (2.12), and also the same initial condition. Thus we conclude, $\bar{A}_n(q) = \bar{Y}_n(q)$. \square

Corollary 3.3. *Let $q = q_c \neq 0$ be a zero of $\overline{A}_n(q)$. Then,*

1. q_c is not a zero for $\overline{A}_n(q)$ whenever $n \neq k$;
2. The function $Q\left(1, 0; \frac{q}{1+q^2}\right)$ has a pole at $q = q_c$.

Proof. This follows from the previous lemma and Lemmas 2.5 and 2.6. □

As in the case of step set \mathcal{S} , we have that $tQ(1, 0) = \sum (-1)^n A_n(1; t) A_n(1; t)$ has an infinite number of singularities and so is not holonomic. Thus consequently $Q(x, y)$, is not holonomic (since one is a specialisation of the other). Thus, already we have proven the first part of Theorem 1.2.

In order to prove that the counting function $W(t) = Q(1, 1)$ is also not holonomic, we must show that the singularities of $Q(1, 0)$ are singularities of $W(t)$ and so are not cancelled out by the singularities of $Q(0, 1)$. This is what we do next.

3.4 The singularities do not cancel

We remark that we can deduce an expression for $B_n B_{n+1}$ similar in spirit to that of Eq. (2.13). Thus, for all points which are not zeroes of some $\overline{B}_n(q)$, the sum $\sum_{k \geq 0} (-1)^k B_k B_{k+1}$ converges.

Now, the singularities for $Q(1, 0)$ and $Q(0, 1)$ have different sources, and thus *a priori* we expect that they will not be the same. We have not been able to do this. However, it is sufficient (and easier) to show that there is an infinite subset of singularities that do not cancel. In particular we consider the singularities of the A_n that lie on the imaginary axis.

For even n we will show that A_n has poles along the imaginary axis. Now B_n has any pole along the imaginary axis, and thus $Q(1, 1)$ has an infinite number of singularities along the imaginary axis. We prove this technical lemma now, and with that our work is done.

Lemma 3.4. *The functions $\overline{A}_n(q)$ and $\overline{B}_n(q)$ satisfy the following recurrences:*

$$\begin{aligned} \overline{A}_n &= \left(\frac{1+q^2}{q}\right) \overline{A}_{n-1} - \overline{A}_{n-2}, & (3.12) \\ \overline{A}_1 &= \frac{4q}{q^2 + 1 + \sqrt{q^4 - 6q^2 + 1}} & \overline{A}_0 = 1 \\ \overline{B}_n &= \left(\frac{1+q^2}{q}\right) \overline{B}_{n-1} - \overline{B}_{n-2} - 1, \\ \overline{B}_1 &= \frac{2q}{q^2 - q + 1 + \sqrt{q^4 - 2q^3 - q^2 - 2q + 1}} & \overline{B}_0 = 1. \end{aligned}$$

Thus, it follows

- (1) For $n \geq 1$, \overline{A}_{2n} has at least two zeros along the imaginary axis;

(2) For any n , and any $q = ir$ along the imaginary axis the value of \overline{B}_n is non-zero, except at $q = 0$.

Proof. Equation (3.12) is a direct consequence of Equation (3.9). (1) From the proof of Lemma 2.5 in Section 2 we see that the zeros of \overline{Y}_n are given by the zeros of the following equation

$$(1 - q^{2n})\sqrt{1 - 6q^2 + q^4} + (1 - q^2)(1 + q^{2n}) = 0. \quad (3.13)$$

By Lemma 3.2 we know that the zeros of \overline{A}_n satisfy the same equation.

Now substitute $q = ir$ and $n = 2m$ into this equation to obtain

$$(1 - r^{4m})\sqrt{1 + 6r^2 + r^4} + (1 + r^2)(1 + r^{4m}) = 0. \quad (3.14)$$

Denote the left-hand side of this equation by $f(r)$, which is a continuous function for real r .

At $r = \pm 1$ we find that $f(\pm 1) = 4$. At $r = \pm 2$ we find that

$$f(\pm 2) = (\sqrt{41} + 5) + (5 - \sqrt{41})2^{4m} \quad (3.15)$$

which is negative for $m \geq 1$. Hence $f(r)$ has at least two zeros, one in the interval $[-2, -1]$ and the other in the interval $[1, 2]$. So \overline{A}_{2n} has zeros on the imaginary axis.

(2) First, we solve the recurrence and give an explicit form for \overline{B}_n in terms of \overline{B}_1 , denoted here by β :

$$\overline{B}_n = \frac{q^{2n}(1 - 2q + \beta q^2 - \beta q) + q^n(q^2 + q) + (q^3 - 2q^2 + q\beta - \beta q^2)}{(-1 + q)^2 (q + 1) q^n}. \quad (3.16)$$

We can now repeat the method used in the proof of Lemma 2.5 to show that the zeros of \overline{B}_n satisfy the following equation:

$$0 = q^2(q^2 - 3q + 1)(q^{4n} + 1) - q(q + 1)^2(q^2 - 3q + 1)(q^{3n} + q^n) - (q^6 - 4q^5 + 3q^4 + 6q^3 + 3q^2 - 4q + 1)q^{2n}. \quad (3.17)$$

Note that this equation is symmetric in $q \leftrightarrow 1/q$ (as we would expect, since we introduced the q variable by substituting $t \mapsto 1/(q + 1/q)$).

We wish to show that the solutions of this equation do not lie on the imaginary axis and so the singularities of A_n which lie on the imaginary axis cannot be cancelled by singularities of B_n . Hence we substitute $q = ir$ with $r \in \mathbb{R}$:

$$0 = r^2(r^2 + 3ir - 1)(r^{4n} + 1) - ir(r - i)^2(r^2 + 3ir - 1)((ir)^{3n} + (ir)^n) + (r^6 + 4ir^5 - 3r^4 + 6ir^3 + 3r^2 + 4ir - 1)r^{2n}(-1)^n. \quad (3.18)$$

Let $\nu(r)$ denote the right-hand side of this expression. Note that $\nu(0) = 0$ for all n and also $\nu(\pm 1) = \pm 8I$, so that $r = 0$ is always a zero and $r = \pm 1$ are never zeros.

We now analyse the real zeros of this expression for different values of $n \pmod 4$. In each case we show that there are no real values for r which annihilate ν (except $r = 0$). We do this by considering the real and imaginary parts of ν separately.

Case 1: $n \equiv 2 \pmod{4}$

In the case that $n \equiv 2 \pmod{4}$, the imaginary part of ν is

$$\Im(\nu) = 3r^{4n+3} + r(r^4 + 4r^2 + 1)r^{3n} + 2r(2r^4 + 3r^2 + 2)r^{2n} + r(r^4 + 4r^2 + 1)r^n + 3r^3. \quad (3.19)$$

This expression is strictly positive for $r > 0$. Since it contains all odd powers of r , it is strictly negative for $r < 0$. It can only be zero when $r = 0$.

Case 2: $n \equiv 0 \pmod{4}$

In this case, ν has real part equal to

$$\Re(\nu) = r^2(r^2 - 1)(r^{4n} + r^{3n} + (r^2 - 2 + r^{-2})r^{2n} + r^n + 1). \quad (3.20)$$

Suppose $r \neq 0, \pm 1$. Then

$$r^2(r^{4n} + r^{3n} + r^n + r + (r^2 - 1)^2 r^n) = 0 \quad (3.21)$$

implies that $r^{2n} + r^n + r^2 + r^{-2} + r^{-n} + r^{-2n} = 2$. This equation is symmetric in $r \leftrightarrow 1/r$ and $r \leftrightarrow -r$ (since n is even). If $r \geq 1$, the left-hand side is strictly larger than 2, and hence has no solution. By symmetries, there can be no real solution.

So the only points that annihilate the real part of ν are $r = 0$ and $r = \pm 1$. As noted above $r = 0$ is a zero of ν , but $r = \pm 1$ are not.

Case 3: n odd

To show that ν has no real zeros (aside from $r = 0$) when n is odd, we consider a linear combination of the real and imaginary parts of ν that eliminates the leading powers of r which in turn simplifies the analysis.

In particular consider $(r - 1/r)\Im(\nu) - 3\Re(\nu)$; this eliminates the $(r^2 + 3ir - 1)(r^{4n} + 1)$ term from equation (3.18). If there exists a real zero of ν then it must also annihilate this linear combination. We will show that no such value of r exists, and so there are no real zeros of ν (excepting $r = 0$).

The linear combination simplifies to

$$\begin{aligned} (r - 1/r)\Im(\nu) - 3\Re(\nu) \\ = r^{2k+2}(2 + 14r^2 + 2r^4)(r^{4k+2} - 1)(-1)^k - (r^2 - 1)(r^4 + 12r^2 + 1)r^{4k+2}. \end{aligned} \quad (3.22)$$

Now divide both sides by $r^2 - 1$ to obtain (we know that $r \neq \pm 1$ are not zeros of ν):

$$\begin{aligned} \frac{1}{r}\Im(\nu) - \frac{3}{r^2 - 1}\Re(\nu) \\ = r^{2k+2}(2 + 14r^2 + 2r^4) \left(\frac{r^{4k+2} - 1}{r^2 - 1} \right) (-1)^k - (r^4 + 12r^2 + 1)r^{4k+2}. \end{aligned} \quad (3.23)$$

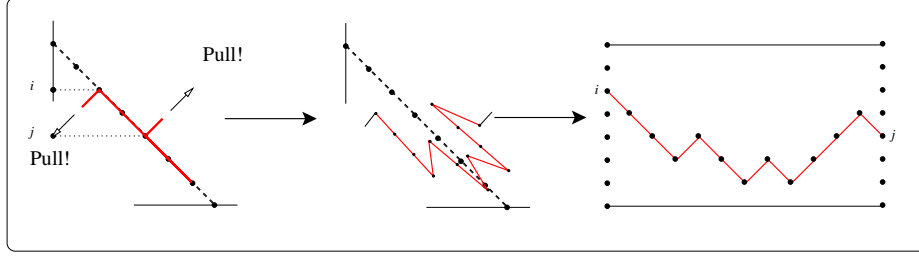


Figure 2: Stretching the walk to find a directed path in a strip

When k is odd this expression is a polynomial in r^2 with negative coefficients, while when k is even this is a polynomial in r^2 with positive coefficients. Hence this is non-zero except at $r = 0$.

Thus, as no \overline{B}_n has a zero on the imaginary axis, (2) holds. \square

It seems clear that $Q(0, 1; t)$ is also non-holonomic. Indeed, the walks are very similar to the symmetric case. However, we were unable to construct a direct argument to show that it too is non-holonomic.

4 Conclusions

4.1 How robust is this method at detecting non-holonomy?

This is a natural question, and it speaks to the ability of this approach to be applied to other problems. In all the cases known to give a holonomic generating function, the group of kernel iterates is finite (of order 8 or less). If one attempts to repeat the proofs of Theorems 2.2 and 3.1 then one obtains a finite set equations. It is not clear that solution of these equations has singularities appearing in the same way as occurred here.

4.2 Combinatorial intuitions of non-holonomy

More generally, we are interested in developing an intuition about when precisely a combinatorial object will have a non-holonomic generating function. If we decompose these walks according to their NE or N steps, a source of singularities becomes apparent. Unfortunately, since we are unable to prove that the singularities do not cancel, the details we present in this section do not constitute a proof of non-holonomy, however they are nonetheless instructive for understanding a potential source of the singularities.

Any walk from either of these two families can be decomposed into a sequence of directed paths in a strip of unrestricted length where after each NE/N step the size of the height bound increases. Figure 2 for an example of such a decomposition for step set \mathcal{S} . Thus, a potential strategy groups lattice walks that end on the line $x + y = k$, and decomposes any such walk into a triple: A walk that ends on the line $x + y = k - 2$, a NE step, and a directed path in a strip of height k .

To describe the generating function, we use Example 11 of [5] who present a generating function for paths of length n in strip of height k of walks that begin at a given height, and end at a given height.

Define $D_k(y; t)$ as the generating function for the subset of walks ending on $x+y = k$ where y marks the final height of the walk. We can easily translate the above decomposition into a functional equation for the generating function. If we then make the substitution $t \mapsto \frac{q}{1+q^2}$, the expression simplifies remarkably into the following recurrence for $D_k(y) = D_k(y, \frac{q}{1+q^2})$:

$$D_k(y) = \frac{q^3 D_{k-2}(q)(y^{k+2} + 1) - qy^2 D_{k-2}(y)(q^{k+2} + 1)}{(q^{k+2} + 1)(yq - 1)(y - q)}. \quad (4.1)$$

In fact, for our purposes it suffices to consider:

$$D_k(1) = \frac{q(q^{k+2} + 1)D_{k-2}(1) - 2q^3 D_{k-2}(q)}{(q^{k+2} + 1)(q - 1)^2}. \quad (4.2)$$

From this formula, and from computations for various values of k , $D_k(1)$ is a rational function in q , and it seems clear that the set (taken over all k) of poles of $D_k(1)$ is dense in the unit circle. Were this so, we would apply the following theorem to the generating function $Q_S(s, s; \frac{q}{1+q^2}) = \sum D_k(1; \frac{q}{1+q^2})s^k$, and thus conclude the non-holonomy of $Q_S(x, y; t)$.

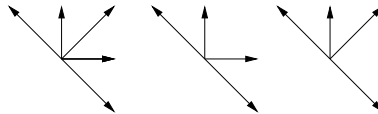
Theorem 4.1. *Let $f(x; t) = \sum_n c_n(x)t^n$ be a holonomic power series in $\mathbb{C}(x)[[t]]$. with rational coefficients in x . For $n \geq 0$ let S_n be the set of poles of $c_n(y)$, and let $S = \bigcup S_n$. Then S has only a finite number of accumulation points.*

Again, the principle difficulty is showing that the singularities do not cancel; that solutions to $q^{k+2} + 1$ are indeed poles of D_k .

This approach was pioneered by Guttman and Enting [7], and has been fruitful for several different models [6, 12]. Unfortunately their arguments do not appear to work here.

4.3 Related walks

We expect the following walks to have non-holonomic generating functions because the groups of their kernel iterates are infinite. It is even likely this can be proved in the same manner as Theorems 1.1 and 1.2.



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References

- [1] Cyril Banderier and Philippe Flajolet. Basic analytic combinatorics of directed lattice paths. *Theoret. Comput. Sci.*, 281(1-2):37–80, 2002. Selected papers in honour of Maurice Nivat.
- [2] Mireille Bousquet-Melou. Walks in the quarter plane: Kreweras’ algebraic model. *Annals of Applied Probability*, 15(2):1451–1491, 2005.
- [3] Mireille Bousquet-Mélou. Rational and algebraic series in combinatorial enumeration. In *International Congress of Mathematicians 2006*, 2006.
- [4] Mireille Bousquet-Mélou and Marko Petkovšek. Walks confined in a quadrant are not always D-finite. *Theoret. Comput. Sci.*, 307(2):257–276, 2003. Random generation of combinatorial objects and bijective combinatorics.
- [5] Philippe Flajolet and Robert Sedgewick. Analytic combinatorics: functional equations, rational and algebraic functions. *INRIA Research Report*, 4103, January 2001.
- [6] A.J. Guttmann. Indicators of solvability for lattice models. *Discr. Math.*, 217:167–189, 2000.
- [7] A.J. Guttmann and I.G. Enting. On the solvability of some statistical mechanical systems. *Phys. Rev. Letters*, 76:344–347, 1996.
- [8] Leonard Lipshitz. D-finite power series. *J. Algebra*, 122(2):353–373, 1989.
- [9] Marni Mishna. Classifying lattice walks in the quarter plane. *preprint: ArXiv:math.CO/0611651*, 2006.
- [10] Helmut Prodinger. The kernel method: a collection of examples. *Sém. Lothar. Combin.*, 50:Art. B50f, 19 pp. (electronic), 2003/04.
- [11] T. Prellberg, A. Rechnitzer, and E. J. Janse van Rensburg. Partially directed paths in a wedge. *preprint arxiv.org:math.CO/0609834*,
- [12] A. Rechnitzer. Haruspicy 2: The anisotropic generating function of self-avoiding polygons is not d-finite. *J. Combinatorial Theory - Series A*, 113:520–546, 2006.
- [13] R. P. Stanley. Differentiably finite power series. *European J. Combin.*, 1(2):175–188, 1980.