7 Problem Set 7 — Complex plane

1. Consider the logistic map $f_\mu(z) = \mu z(1-z)$.

   (a) Find the region $\mu \in \mathbb{C}$ such that $f_\mu(z)$ has an attracting fixed point.

   - The fixed points are solutions of $f(z) - z = 0$ which are
   $$z = 0 \qquad z = (\mu - 1)/\mu$$
   - The derivative of $f(z)$ is
   $$f'(z) = \mu(1 - 2z)$$
   - The fixed point at $z = 0$ is stable if
   $$|f'(0)| = |\mu| < 1$$
   which defines a circle of radius 1 centred at $\mu = 0$ in the complex μ-plane.
   - The fixed point at $z = (\mu - 1)/\mu$ is stable if
   $$|f'(\frac{\mu-1}{\mu})| = |2 - \mu| < 1$$
   which defines a circle of radius 1 centred at $\mu = 2$ in the complex μ-plane.

   (b) Find the region $\mu \in \mathbb{C}$ such that $f_\mu(z)$ has an attracting 2-cycle.

   - The 2-cycle is the solution of $f(f(z)) - z = 0$. These are:
   $$z = 0, \frac{\mu-1}{\mu}, \frac{1}{2\mu}\left(\mu + 1 \pm \sqrt{(1+\mu)(\mu-3)}\right)$$
   - Of these, the first 2 are the fixed points, while the last two are the 2-cycle — call them $p_\pm$.
   - To work out where this is stable we need to find for which μ values:
   $$|f'(p_-)f'(p_+)| = \mu^2|(1-2p_-)(1-2p_+)| < 1$$
   - Expanding this gives
   $$(1-2p_-)(1-2p_+) = 1 - 2(p_+ + p_-) + 4(p_+p_-)$$
   - Computing each bit:
   $$p_- + p_+ = \frac{1+\mu}{\mu}$$
   and
   $$\begin{aligned} p_-p_+ &= \frac{1}{4\mu^2}\left((1+\mu)^2 - (1+\mu)(\mu-3)\right) \\ &= \frac{1+\mu}{4\mu^2}(1 + \mu + 3 - \mu) = \frac{1+\mu}{\mu^2} \end{aligned}$$

- Putting these into the expression above gives:

$$\left|\mu^2\left(1-2\frac{1+\mu}{\mu}+4\frac{1+\mu}{\mu^2}\right)\right|<1$$

which simplifies to:

$$|4+2\mu-\mu^2|<1$$

- Now we need to do some work — lets find the boundary

$$4+2\mu-\mu^2=e^{i\theta}$$

This gives

$$\mu=1\pm\sqrt{5-e^{i\theta}}$$

- This gives two curves in the μ-plane — unfortunately they don't simplify further — they are almost circles.

2. Consider the quadratic map $Q_c(z)=z^2+c$.

(a) Find the slope of Q_c at the (stable) fixed point (as a function of c).

- The slope is $2z_0=1-\sqrt{1-4c}$.

(b) Find the slope of Q_c^2 at the 2-cycle (as a function of c).

- The slope is $4z_1z_2=4c+4$.

(c) An approximate renormalisation scheme for the period doubling of Q_c can be obtained by equating these two slopes. Show that this leads to the relation

$$c_{n-1}=-2-6c_n-4c_n^2$$

where c_n approximates the location of the stable 2^{n-1}-cycle.

- Put the slope of the fp as $1-\sqrt{1-4c_1}$ and the slope of the 2-cycle as $4c_2+4$. Equating these gives:

$$\begin{aligned}
-\sqrt{1-4c_1} &= 4c_2+3\\
1-4c_1 &= 16c_2^2+24c_2+9\\
c_1 &= -2-6c_2-4c_2^2
\end{aligned}$$

If we now assume this to hold between the slope of $Q^{2^{n-1}}$ at the 2^{n-1}-cycle and the slope of Q^{2^n} at the 2^n-cycle then we obtain the above relation.

(d) Show that this leads to an approximation of $c_\infty=-\frac{7+\sqrt{17}}{8}$ — the location of transition to chaos.

- If we assume that the sequence of c_n converges to a fixed point c_∞, then c_∞ satisfies $c_\infty = -2 - 6c_\infty - 4c_\infty^2$. Solving this gives:

$$c_\infty = -\frac{1}{8}(-7 \pm \sqrt{17}) \approx \frac{-3}{8}, \frac{-11}{8}$$

The value $c_\infty = -\frac{1}{8}(-7+\sqrt{17})$ we can discount since the fixed point is stable for this value of c. This leaves the other value.

(e) Show that this also leads to the approximate feigenvalue, $\delta = 1 + \sqrt{17}$.

- Put $c_n = c_\infty + \epsilon_n$, and substitute it into the relation. Some algebra leads to:

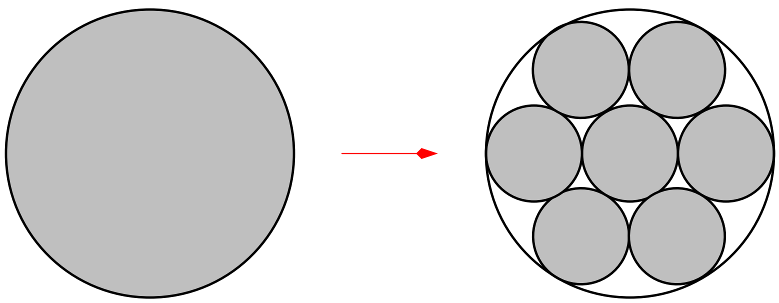
$$\epsilon_{n-1} = \epsilon_n + \sqrt{17}\epsilon_n - 4\epsilon_n^2$$

Ignoring the ϵ^2 terms (since they are very small) gives:

$$\epsilon_{n-1}/\epsilon_n = 1 + \sqrt{17}$$

If we use the scaling form $c_n = c_\infty + A/\delta^n$ then we see that $\epsilon_{n-1}/\epsilon_n = \delta$.

3. Consider the following construction of a fractal "gasket". Start with a circle of radius 1 and remove the region *outside* the 7 circles of radius 1/3. Repeat this procedure for each of the 7 interior circles and so on.



(a) Give the diameter of the circles at the n-th stage.

- At each stage the diameter is reduced by a factor of 3. So the diameter is $2/3^n$.

(b) Give the number of circles at the n-th stage.

- Each circle is replaced by 7 smaller circles at each stage. Hence the number of circles is 7^n.

(c) Calculate the area of the fractal.

- At the n-th stage there are 7^n circles of radius $1/3^n$. This gives a total area of $7^n \times \pi 3^{2n} = \pi(7/9)^n$. Hence the area goes to zero.

(d) Calculate the fractal dimension of the object.

- Each circle of radius r may be covered by a square of side length $2r$. Hence at the n-th stage we require 7^n squares of side-length $2/3^n$.

$$7^n = A \times 2 \times 3^{nD}$$

Hence the fractal dimension D is $\log 7/\log 3 \approx 1.771243749\ldots$.

4. Completely describe the orbits of the following 2-dimensional system:

$$\mathbf{x}_{n+1} = \begin{pmatrix} -4 & 3 \\ 5 & -1/2 \end{pmatrix} \mathbf{x}_n$$

(including stable and unstable manifolds).

- The system is expansive since the determinant is -13.
- The eigenvalues and eigenvectors are:

$$\lambda_1 = 2 \qquad \mathbf{v}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$\lambda_2 = -13/2 \qquad \mathbf{v}_1 = \begin{pmatrix} -6/5 \\ 1 \end{pmatrix}$$

- There is no stable manifold. The unstable manifold is the space spanned by $\{\mathbf{v}_1, \mathbf{v}_2\}$ which is all of $\mathbb{R}^2$.
- Hence the point $\mathbf{x} = \mathbf{0}$ is an unstable fixed points and the orbits of all other points are repelled from it.
- Along the line $y = 2x$ points are multiplied by 2 at each iteration. Points on this line are repelled from $\mathbf{0}$.
- Along the line $y = -5x/6$, points are multiplied by $-13/2$ at each iteration. Hence orbits along this second line "bounce" on either side of the origin while being repelled.
- This gives rise to a phase portrait something like: