4 Problem Set 4 — Bifurcations

1. Each of the following functions undergoes a bifurcation at the given parameter value. In each case use analytic or graphical techniques to identify the type of bifurcation (saddle node or period doubling or neither). Also sketch a "typical" phase portrait for values of the parameter above, at and below the indicated value.

   (a) $F_\lambda(x) = x + x^2 + \lambda$ at $\lambda = 0$

   - F_λ has two fixed points at $x = \pm\sqrt{-\lambda}$. Hence there are no fixed points for $\lambda > 0$. $F' = 2x + 1$, so there is a neutral fixed point at $x = 0$ for $\lambda = 0$. For $\lambda < 0$, $x = +\sqrt{-\lambda}$ is a repelling fixed point. For $-1 < \lambda < 0$, $x = -\sqrt{-\lambda}$ is an attracting fixed point. Hence this is a saddle-node bifurcation.

   (b) $F_\lambda(x) = x + x^2 + \lambda$ at $\lambda = -1$

   - Continuing the previous question, we see that at $\lambda = -1$, the fixed point at $x = -\sqrt{\lambda} = -1$ becomes neutral with derivative $= -1$. This suggests a period doubling bifurcation. Indeed we can check (if we get the algebra right):

     $$F(F(x)) - x = (\lambda + x^2 + 2x + 2)(\lambda + x^2).$$

     This gives the fixed points at $x = \pm\sqrt{-\lambda}$, and also the location of a two-cycle at $x = -1 \pm \sqrt{-1-\lambda}$. This two-cycle is born at the neutral fixed point when $\lambda < -1$. Hence this is a period doubling bifurcation.

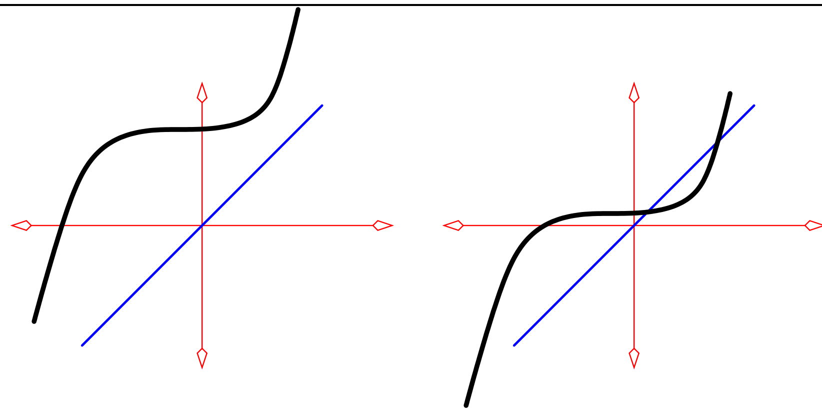
   (c) $S_\mu(x) = \mu \sin x$ at $\mu = 1$

   - For $-1 < \mu < 1$, a plot of $S(x)$ shows that there is only 1 fixed point at $x = 0$. For $-1 < \mu < 1$ it is attracting, while for $|\mu| > 1$ it is repelling. For $\mu > 1$ two new fixed points are created.

   (d) $S_\mu(x) = \mu \sin x$ at $\mu = -1$

   - Continuing the previous question, we see that the origin becomes a neutral fixed point at $\mu = -1$, and that the derivative at $x = 0$ is -1. This suggests a period doubling bifurcation. Again, a careful plot of $S(S(x))$ shows that a two-cycle is created and that it is attracting.

   (e) $F_c(x) = x^3 + c$ at $c = 2/3\sqrt{3}$



- Careful plotting is required for this one. We see that for "large" c, F has only one intersection with $y = x$ and this occurs for $x < 0$. Whereas for "small" c, F has three intersections with $y = x$, the two new intersections occur for $x > 0$ — this suggests a saddle-node bifurcation. Indeed we can check that at $c = 2/3\sqrt{3}$ that $F(x) = x$ has two solutions, one of which corresponds to a neutral fixed point:

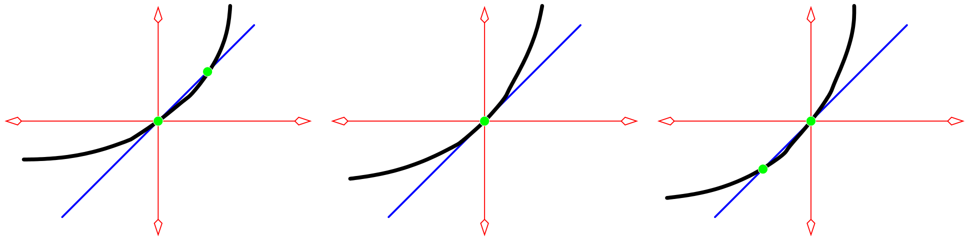
$$x^3 - x + 2/3\sqrt{3} = (x + 2/\sqrt{3})(x - 1/\sqrt{3})^2.$$

(You can find this by looking for factorisations of the form $(x - a)(x - b)^2$). Checking the derivative at $x = 1/\sqrt{3}$ shows that this point is a neutral fixed point. Hence this is a saddle node bifurcation.

(f) $E_\lambda(x) = \lambda(e^x - 1)$ at $\lambda = -1$

- A careful plot shows that there is a fixed point at $x = 0$, and that this is the only one. The derivative at this fixed point is simple $E'(0) = \lambda$. Hence at $\lambda = -1$ we expect that there is a period doubling bifurcation. A careful plot will verify this.

(g) $E_\lambda(x) = \lambda(e^x - 1)$ at $\lambda = 1$



- From the previous question we see that the fixed point at $x = 0$ becomes neutral at $\lambda = 1$, with derivative 1. For $\lambda < 1$ there is a second fixed point > 0, while for $\lambda > 1$ there is a second fixed point < 0.

The following questions (2–9) deal with the logistic equation $F_\lambda(x) = \lambda x(1 - x)$.

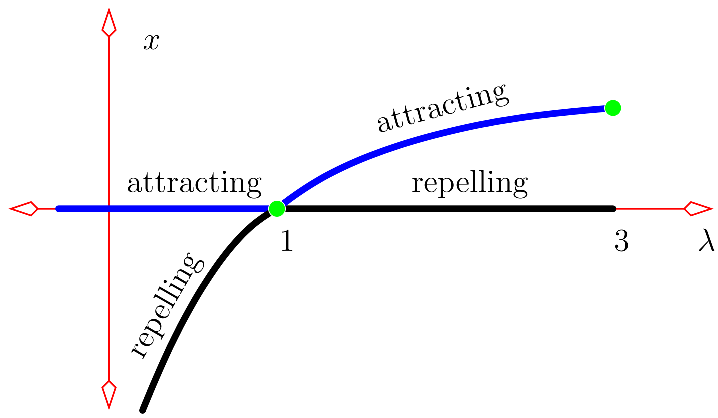
2. For which values of λ does F_λ have an attracting fixed point at $x = 0$?

3. For which values of λ does F_λ have a non-zero attracting fixed point?

4. Describe the bifurcation that occurs at $\lambda = 1$.

5. Sketch the phase portrait and bifurcation diagram near $\lambda = 1$.

   - Let us first find the location of the fixed points of F:
   $$F(x) - x = \lambda x(1-x) - x = x(\lambda - 1 - \lambda x)$$
   Hence there are fixed points at $x = 0$ and $x = \frac{\lambda-1}{\lambda}$. The derivative of F is $F' = \lambda(1-2x)$. Hence the fixed point at $x = 0$ is attracting for $|\lambda| < 1$.
   - At the other fixed point shows $F'(\frac{\lambda-1}{\lambda}) = 2-\lambda$. Hence this fixed point is attracting for $1 < \lambda < 3$.
   - For $\lambda \neq 1$ there are two fixed points. For $\lambda < 1$ the fixed point at $x = 0$ is attracting, and it becomes neutral when it coalesces with the other fixed point when $\lambda = 1$. The non-zero fixed point then becomes attracting for $\lambda > 1$.



6. Describe the bifurcation that occurs at $\lambda = 3$.

7. Sketch the phase portrait and bifurcation diagram near $\lambda = 3$.

   - When $\lambda = 3$, the fixed point at $x = \frac{\lambda-1}{\lambda}$ becomes neutral with derivative $= -1$. This suggests a period doubling bifurcation. Solving $F(F(x)) - x = 0$ gives:
   $$\begin{aligned} F(F(x)) - x &= \lambda\big(\lambda x(1-x)\big)\big(1 - \lambda x(1-x)\big) - x \\ &= \text{some algebra} \\ &= \Big((\lambda-1)x - \lambda x^2\Big)\Big(\lambda^2 x^2 - \lambda(\lambda+1)x + (\lambda+1)\Big) \end{aligned}$$
   Note — we use the fact that the fixed points of $F(x)$ must also be fixed points of $F(F(x))$ to help us factorise the quartic polynomial. This tells us that $\big((\lambda - 1)x - \lambda x^2\big)$ must be a factor (since this is the polynomial we had to solve to find the fixed points of $F(x)$).

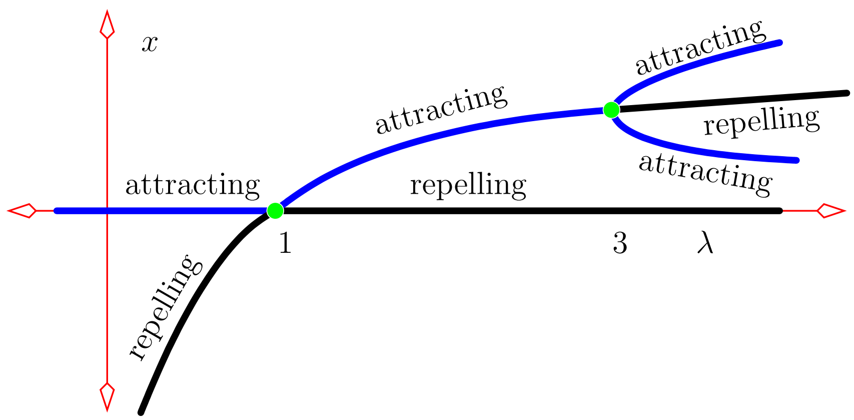
- Solving the second quadratic polynomial will give us the location of the 2-cycle:

$$q_{\pm} = \frac{1}{2\lambda}\left(\lambda + 1 \pm \sqrt{(\lambda+1)(\lambda-3)}\right)$$

Hence this 2-cycle only exists when $\lambda > 3$ or $\lambda < -1$. Some messy algebra shows that

$$F'(q_-)F'(q_+) = 4 + 2\lambda - \lambda^2$$

This then shows that the two cycle is attracting for $3 < c < 1 + \sqrt{6} \approx 3.449$.



8. Describe the bifurcation that occurs at $\lambda = -1$.

9. Sketch the phase portrait and bifurcation diagram near $\lambda = -1$.

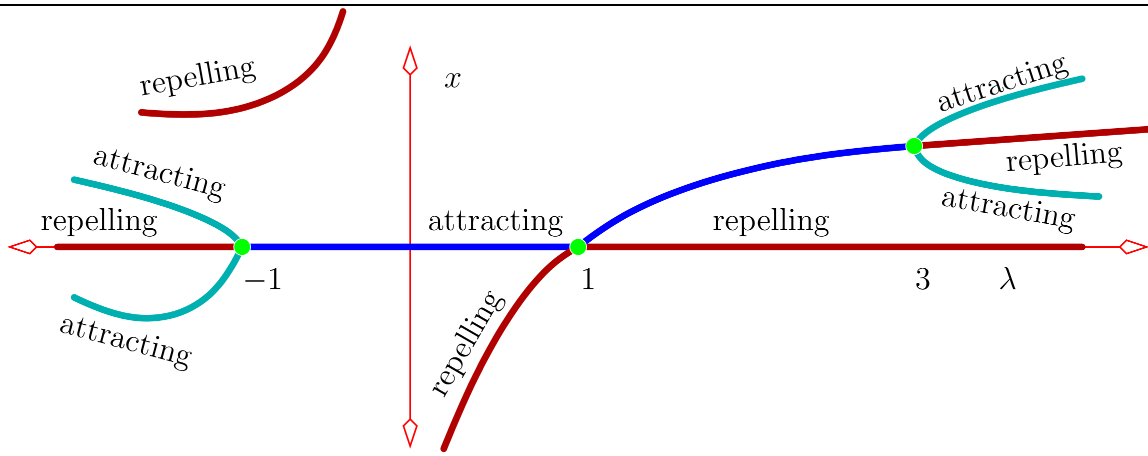
- The attracting fixed point at $x = 0$ becomes neutral at $\lambda = -1$, and from the above workings we see that a 2-cycle is born when $\lambda < -1$. Again, this 2-cycle is given by:

$$q_{\pm} = \frac{1}{2\lambda}\left(\lambda + 1 \pm \sqrt{(\lambda+1)(\lambda-3)}\right)$$

and so its stability is again determined by

$$|F'(q_-)F'(q_+)| = |4 + 2\lambda - \lambda^2| < 1$$

Since we are now interested in $\lambda < -1$, this equation now tells us that the 2-cycle is stable for $1 - \sqrt{6} < \lambda < -1$.



10. Consider $F_\lambda = \lambda x - x^3$. Show that the 2-cycle given by $\pm\sqrt{\lambda+1}$ is repelling when $\lambda > -1$.

- In order to show that a two cycle $q_\pm$ is repelling, we need to show that $|F'(q_+)F'(q_-)| > 1$. $F' = \lambda - 3x^2$, so:

$$\begin{aligned} F'(q_+)F'(q_-) &= \left(\lambda - 3q_+^2\right)\left(\lambda - 3q_-^2\right) \\ &= \left(\lambda - 3(\lambda+1)\right)^2 \\ &= (3+2\lambda)^2 \end{aligned}$$

Since this is a square, it is never negative. It is equal to 1 when

$$(3+2\lambda)^2 = 1 \longrightarrow 4\lambda^2 + 12\lambda + 8 = 0 \longrightarrow (2\lambda+4)(2\lambda+2) = 0$$

i.e. when $\lambda = -2, -1$. It is bigger than 1 when

$$\infty < c < -2 \qquad \text{or} \qquad -1 < c < \infty$$

Hence the two-cycle is repelling for $c > -1$.

11. Consider the family of functions $F_\lambda(x) = x^5 - \lambda x^3$. Discuss the bifurcation of 2-cycles that occurs when $\lambda = 2$. Note that this function is an odd function of x for all λ — so points of period 2 can be found by solving $F_\lambda(x) = -x$.

- We find the 2-cycles by solving $F(x) = -x$:

$$F(x) + x = x^5 - \lambda x^3 + x = x(x^4 - \lambda x^2 + 1)$$

This then has solutions:

$$\begin{aligned} x &= 0 \\ x^2 &= \frac{1}{2}\left(\lambda \pm \sqrt{\lambda^2 - 4}\right) \end{aligned}$$

Now, $x = 0$ is a fixed point, so the other points form the two-cycles:

$$\begin{aligned} p_{\pm} &= \pm\frac{1}{2}\sqrt{2\lambda + 2\sqrt{\lambda^2 - 4}} \\ q_{\pm} &= \pm\frac{1}{2}\sqrt{2\lambda - 2\sqrt{\lambda^2 - 4}} \end{aligned}$$

(since $F(\pm x) = \mp x$). We see that the 2-cycles do not exist for $\lambda < 2$. We might expect that there is a period doubling bifurcation at $\lambda = 2$ since there are 2-cycles involved. Let us set $\lambda = 2$ and look at the locations of fixed points and 2-cycles:

$$F(x) - x = x^5 - 2x^3 - x = x(x^4 - 2x^2 - 1)$$

So the fixed points are at $x = 0, \pm\sqrt{1 + \sqrt{2}}, \pm i\sqrt{-1 + \sqrt{2}}$. Setting $\lambda = 2$ in the above expressions for the 2-cycles gives:

$$\begin{aligned} p_{\pm} &= \pm 1 \\ q_{\pm} &= \pm 1 \end{aligned}$$

And so the 2-cycles do not coalesce with the fixed points at $\lambda = 2$ as we might expect with a period doubling bifurcation. Instead this is an example of a saddle-node bifurcation of a 2-cycle.