1 Solutions to problem set 1

1. Let $F(x) = x^2$. Compute the first five points of the orbit of $1/2$.
   - $\{x_0, \ldots, x_5\} = \{2^{-1}, 2^{-2}, 2^{-4}, 2^{-8}, 2^{-16}, 2^{-32}\}$
2. Let $F(x) = x^2 - 1$. Compute $F^2(x)$ and $F^3(x)$.
   - $F^2(x) = (x^2 - 1)^2 - 1 = x^4 - 2x^2$
   - $F^3(x) = (x^4 - 2x^2)^2 - 1 = x^8 - 4x^6 + 4x^4 - 1$
3. Find all the real fixed points of the following functions
   (a) $F(x) = 3x + 2$
      - $x = -1$.

   (b) $F(x) = x^2 - 2$
      - $x = 2, -1$.

   (c) $F(x) = x^3 - 3x$
      - $x = 0, \pm 2$.

   (d) $F(x) = |x|$
      - $\{x \mid x \in \mathbb{R}, x \geq 0\}$.

   (e) $F(x) = x \sin x$
      - $x = \frac{1}{2}\pi + 2\pi k$ where $k \in \mathbb{Z}$.
4. Find the fixed points and two-cycles of the function $F(x) = 1 - x^2$.
   - Fixed points are $x = \frac{1}{2}\left(-1 \pm \sqrt{5}\right)$.
   - Two cycle is $x = 0, 1$.

The following questions correspond to the *doubling map*,

$$D : [0, 1) \mapsto [0, 1)$$

defined by

$$D(x) = \begin{cases} 2x & 0 \leq x < 1/2 \\ 2x - 1 & 1/2 \leq x < 1 \end{cases}$$

which is equivalent to

$$D(x) = 2x \mod 1.$$

5. Discuss the orbits of the following points under $D(x)$:

   (a) $x_0 = 0.3$

   - Is eventually periodic, with period 4. Cycle is $\{0.6, 0.2, 0.4, 0.8\}$.

   (b) $x_0 = 0.7$

   - Is eventually periodic, with period 4. Cycle is $\{0.6, 0.2, 0.4, 0.8\}$.

   (c) $x_0 = 1/8$

   - Is eventually fixed at $x = 0$.

   (d) $x_0 = 1/7$

   - Is periodic with period 3. Cycle is $\{1/7, 2/7, 4/7\}$.

   (e) $x_0 = 3/11$

   - Is periodic with period 10. Cycle is $x_i/11$, where x_i is the i^{th} element of $\{3, 6, 1, 2, 4, 8, 5, 10, 9, 7\}$.

6. Explain why a computer might have difficulty computing the orbit of 1/7 if you give a decimal expansion (*Hint* — computers store numbers in binary).

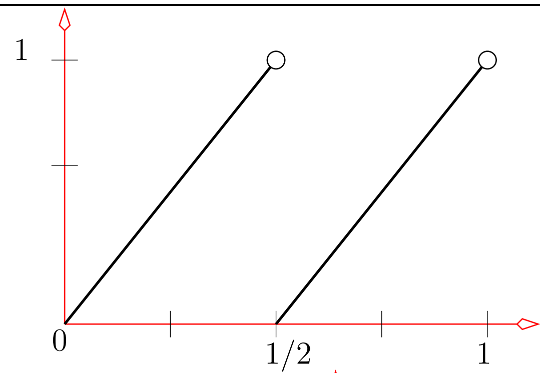
   - A computer stores non-integer numbers by their binary expansion. Typically it stores only the first 50 or so digits of this expansion — causing some rounding error. Applying $D(x)$ to such a binary expansion multiplies the expansion by 2 and cuts off the integer part — which is equivalent to moving the decimal place one place to the right. Consequently the rounding error that occured when the number was stored grows until it is the same size as the number being stored!

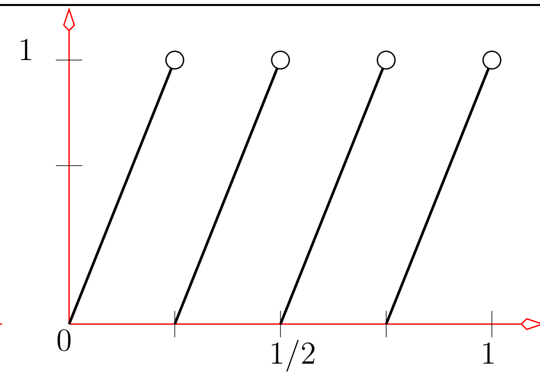
7. Write down an explicit formula for $D^2(x)$. Draw a graph of $D(x)$, $D^2(x)$ and $D^3(x)$.

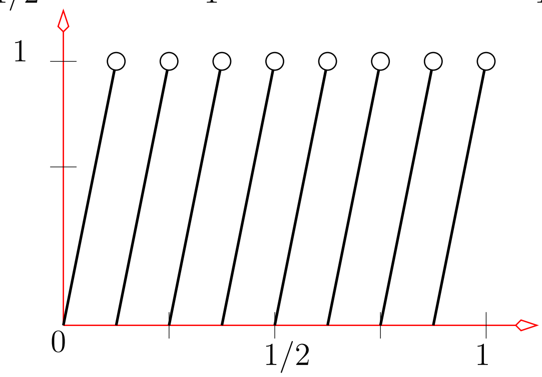
   - $D(x) \equiv 2x \mod 1$, so $D^2(x) \equiv 4x \mod 1$. Alternatively

$$D^2(x) = \begin{cases} 4x & 0 \leq x < 1/4 \\ 4x - 1 & 1/4 \leq x < 1/2 \\ 4x - 2 & 1/2 \leq x < 3/4 \\ 4x - 3 & 3/4 \leq x < 1 \end{cases}$$

   - Plots of $D(x), D^2(x)$ and $D^3(x)$ (respectively):







In general $D^n(x)$ will be made up of 2^n diagonals each of height 1 and width $1/2^n$.

8. Find all the fixed points of $D(x)$, $D^2(x)$ and $D^3(x)$. How many fixed points does $D^n(x)$ have?

   - Drawing a line from $(0,0)$ to $(1,1)$ on the plots of $D(x)$, $D^2(x)$ and $D^3(x)$, we see that it crosses each of the diagonals in the plot, *except* the last one — since the point $y = 1$ is not within the range of $D(x)$. Hence $D^n(x)$ has $2^n - 1$ fixed points.
   - The fixed points of $D(x)$ can be found by solving

     $$2x \mod 1 = x$$

     or

     $$2x - x = k, \qquad k \in \mathbb{Z} \text{ and } x \in [0, 1)$$

     Hence the only fixed point is $x = 0$.
   - The fixed points of $D^2(x)$ can be found by solving

     $$4x \mod 1 = x$$

     or

     $$4x - x = k, \qquad k \in \mathbb{Z} \text{ and } x \in [0, 1)$$

     Hence there are 3 fixed points $0, 1/3, 2/3$.

- The fixed points of $D^3(x)$ can be found by solving

$$8x \mod 1 = x$$

or

$$8x - x = k, \qquad k \in \mathbb{Z} \text{ and } x \in [0, 1)$$

Hence there are 7 fixed points $0, 1/7, 2/7, 3/7, 4/7, 5/7, 6/7$.

The following questions correspond to the *tent map*,

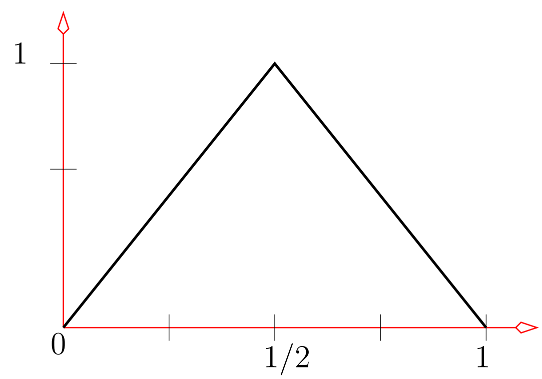
$$T : [0, 1] \mapsto [0, 1]$$

defined by

$$T(x) = \begin{cases} 2x & 0 \le x \le 1/2 \\ 2 - 2x & 1/2 < x \le 1 \end{cases}$$

9. Sketch $T(x)$ — the name of this function should become obvious.

   - The plot is simply a tent-like triangle:

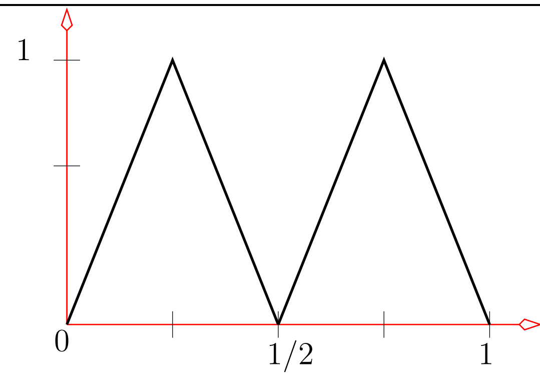


10. Find a formula for $T^2(x)$ and sketch this function.

   - The formula is:

$$T^2(x) = \begin{cases} 4x & 0 \le x \le 1/4 \\ 2 - 4x & 0 < x \le 1/4 \\ 4x - 2 & 1/2 < x \le 3/4 \\ 4 - 4x & 3/4 < x \le 1 \end{cases}$$

   And it looks like:



11. Find all fixed points of $T(x)$ and $T^2(x)$.

    - Drawing the line $y = x$ on the plot of $T(x)$, we see that they intersect at two places — one for each of the segments of the tent. Solving on each of the segments gives the fixed points: $x = 0$ and $x = 2/3$. Similarly the fixed points of $T^2(x)$ are $x = 0, 2/5, 2/3, 4/5$.

12. What does the graph of $T^n(x)$ look like and how many fixed points does it have?

    - The graph of $T^n(x)$ is made up of 2^n "tents" each of which has width $1/2^n$. The line $y = x$ will cross each of the tents exactly twice — so there will be 2^{n+1} fixed points.