7 Problem Set 7 — Complex plane and other stuff

1. Consider the logistic map $f_\mu(z) = \mu z(1-z)$.

   (a) Find the region $\mu \in \mathbb{C}$ such that $f_\mu(z)$ has an attracting fixed point.

   (b) Find the region $\mu \in \mathbb{C}$ such that $f_\mu(z)$ has an attracting 2-cycle. It is enough to show that

   $$|4 + 2\mu - \mu^2| < 1$$

2. Consider the quadratic map $Q_c(z) = z^2 + c$.

   (a) Find the slope of Q_c at the (stable) fixed point (as a function of c).

   (b) Find the slope of Q_c^2 at the 2-cycle (as a function of c).

   (c) An approximate renormalisation scheme for the period doubling of Q_c can be obtained by equating these two slopes. Show that this leads to the relation

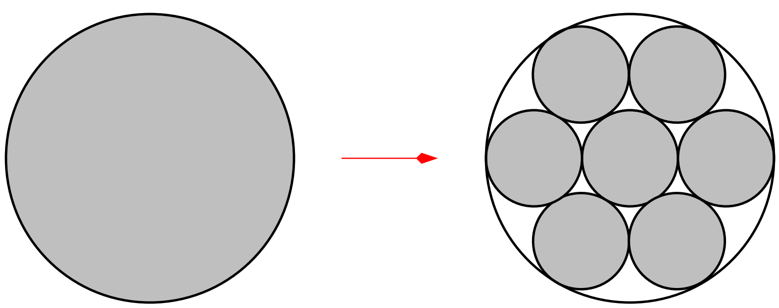
   $$c_{n-1} = -2 - 6c_n - 4c_n^2$$

   where c_n approximates the location of the stable 2^{n-1}-cycle.

   (d) Show that this leads to an approximation of $c_\infty = -\frac{7+\sqrt{17}}{8}$ — the location of transition to chaos.

   (e) Show that this also leads to the approximate feigenvalue, $\delta = 1 + \sqrt{17}$.

3. Consider the following construction of a fractal "gasket". Start with a circle of radius 1 and remove the region *outside* the 7 circles of radius 1/3. Repeat this procedure for each of the 7 interior circles and so on.



   (a) Give the diameter of the circles at the n-th stage.

   (b) Give the number of circles at the n-th stage.

   (c) Calculate the area of the fractal.

   (d) Calculate the fractal dimension of the object.

4. Completely describe the orbits of the following 2-dimensional system:

$$\mathbf{x}_{n+1} = \begin{pmatrix} -4 & 3 \\ 5 & -1/2 \end{pmatrix} \mathbf{x}_n$$

(including stable and unstable manifolds).