6   Problem Set 6 — Symbolic Dynamics

1. Find all points in Σ that are distance exactly $1/2$ from the point $(0000\ldots)$.

2. Find two points halfway between $(000\ldots)$ and $(111\ldots)$. Are there any other such points? Why or why not?

3. Decide whether or not the following sets are dense in $[0, 1]$.

   (a) The set of all numbers in $[0, 1]$ except those of the form $1/2^n$, $n = 1, 2, 3, \ldots$.

   (b) The Cantor middle thirds set.

   (c) The compliment of the Cantor middle thirds set.

4. Is the orbit of the point $(01\ 001\ 0001\ 00001\ \ldots)$ under σ dense in Σ?

The following questions concern the space of sequences on N symbols, Σ_N, together with the shift map σ_N and the distance function:

$$d[s, t] = \sum_{k=0}^{\infty} \frac{|s_k - t_k|}{N^k}$$

5. Prove that $\sigma_N : \Sigma_N \mapsto \Sigma_N$ is continuous.

6. How many points of *prime*-period 2 does σ_N have?

7. Define the new distance function:

$$d_\delta[s, t] = \sum_{k=0}^{\infty} \frac{\delta_k(s, t)}{N^k}$$

where

$$\delta_k(s, t) = \begin{cases} 1 & \text{if } s_k \neq t_k \\ 0 & \text{if } s_k = t_k \end{cases}$$

Prove that $d_\delta[s, t]$ is also a metric on Σ_N.

8. Using $d_\delta[s, t]$ what is the maximum distance between two points in Σ_N?