4 Problem Set 4 — Bifurcations

1. Each of the following functions undergoes a bifurcation at the given parameter value. In each case use analytic or graphical techniques to identify the type of bifurcation (saddle node or period doubling or neither). Also sketch a "typical" phase portrait for values of the parameter above, at and below the indicated value.

   (a) $F_\lambda(x) = x + x^2 + \lambda$ at $\lambda = 0$

   (b) $F_\lambda(x) = x + x^2 + \lambda$ at $\lambda = -1$

   (c) $S_\mu(x) = \mu \sin x$ at $\mu = 1$

   (d) $S_\mu(x) = \mu \sin x$ at $\mu = -1$

   (e) $F_c(x) = x^3 + c$ at $c = 2/3\sqrt{3}$

   (f) $E_\lambda(x) = \lambda(e^x - 1)$ at $\lambda = -1$

   (g) $E_\lambda(x) = \lambda(e^x - 1)$ at $\lambda = 1$

The following questions (2–9) deal with the logistic equation $F_\lambda(x) = \lambda x(1 - x)$.

2. For which values of λ does F_λ have an attracting fixed point at $x = 0$?

3. For which values of λ does F_λ have a non-zero attracting fixed point?

4. Describe the bifurcation that occurs at $\lambda = 1$.

5. Sketch the phase portrait and bifurcation diagram near $\lambda = 1$.

6. Describe the bifurcation that occurs at $\lambda = 3$.

7. Sketch the phase portrait and bifurcation diagram near $\lambda = 3$.

8. Describe the bifurcation that occurs at $\lambda = -1$.

9. Sketch the phase portrait and bifurcation diagram near $\lambda = -1$.

10. Consider $F_\lambda = \lambda x - x^3$. Show that the 2-cycle given by $\pm\sqrt{\lambda + 1}$ is repelling when $\lambda > -1$.

11. Consider the family of functions $F_\lambda(x) = x^5 - \lambda x^3$. Discuss the bifurcation of 2-cycles that occurs when $\lambda = 2$. Note that this function is an odd function of x for all λ — so points of period 2 can be found by solving $F_\lambda(x) = -x$.