Assignment 1

1. Consider the dynamical system on the circle S^1 defined by:

$$\theta_{n+1} = \alpha\, \theta_n.$$

Describe the dynamics of the system for values of $\alpha \geq 0$. Please include discussion of:

- fixed points (and their nature — *i.e.* attracting, repelling or neutral),
- periodic points (and their nature),
- "sensitive dependence on initial conditions",

and anything else you feel is relevant.

Total = 10 marks

- If $\alpha = 0$ then $\theta = 0$ is fixed and all other θ are eventually fixed.

1 mark

- If $0 < \alpha < 1$ then $\theta_n = \alpha^n \theta_0$ for all $n \in \mathbb{Z}^+$. Hence all orbits converge to $\theta = 0$ and $\theta = 0$ is an attracting fixed point.

1 mark

- If $\alpha = 1$ then all $\theta \in S^1$ are fixed.

1 mark

- If $\alpha > 1$ then things are more complicated. For $\alpha \in \mathbb{N}$ things are as I describe them below. For more general α one needs to be quite a bit more careful. I was happy to give full marks for what follows. For those of you that did carefully investigate non-integer α, I generally gave full marks.
- First find fixed points — this is equivalent to solving:

$$\begin{aligned} \alpha\theta &= \theta \mod 1 \\ \alpha\theta - \theta &= k \in \mathbb{Z} \end{aligned}$$

Hence the fixed points are given by $\theta_* = \frac{k}{\alpha-1}$. Since $\theta \in [0, 1)$, it follows that there are $\lfloor \alpha - 1 \rfloor$ (*i.e.* the smallest integer less than $\alpha - 1$) fixed points.
If $\alpha = 3.5$ (for example) the fixed points are $\{0, 1/2.5, 2/2.5\} = \{0, 0.4, 0.8\}$. Since $\theta_{n+1} = \alpha\theta_{n+1}$ the derivative is α at all points, and so all fixed points are repelling.

3 marks

- Find points of period n:

$$\begin{aligned} \alpha^n\theta &= \theta \mod 1 \\ \alpha^n\theta - \theta &= k \in \mathbb{Z} \end{aligned}$$

Hence the period-n periodic points are given by $\frac{k}{\alpha^n-1}$, and so there are $\lfloor \alpha^n - 1 \rfloor$ such points. Since $\theta_{n+1} = \alpha\theta_{n+1}$ the derivative is α at all points, and so all periodic points are repelling.

2 marks

- To see "sensitive dependence on initial conditions" pick two points $x, y \in S^1$ such that $|x_0 - y_0| = \varepsilon > 0$. Under n-applications of the mapping $|x_n - y_n| = |\alpha^n x_0 - \alpha^n y_0| = \alpha^n\varepsilon$. Provided $\alpha > 1$, we can blow up the initial error, ε , by repeated iterations, to make it as big as we want — so it can reach a size comparable with the system size. So two points that start close together will have orbits that diverge by a large amount after some (finite) number of iterations.

2 marks

◁ ◁ ◇ ▷ ▷

2. Consider the dynamical system on the circle S^1 defined by:

$$\theta_{n+1} = 2\,\theta_n.$$

- Prove that the set of all periodic points of this system is dense in the circle S^1.
- Also prove that the set of points that are *not* eventually periodic is also dense in S^1.

Total = 5 marks

- We first have to find all the periodic points. This means we find have to solve

$$\theta_n = \theta_0$$

which is the same as solving:

$$\begin{aligned} 2^n\theta &= \theta \mod 1 \\ 2^n\theta - \theta &= k \in \mathbb{Z} \\ \theta = \frac{k}{2^n - 1}. \end{aligned}$$

Since $\theta \in [0, 1)$, the variable k takes the values $\{0, 1, \ldots, 2^n - 1\}$.

1 mark

- The points of period n are evenly spaced around the circle, and so they partition the circle into arcs of length $1/(2^n - 1)$. If we pick any two points on the circle $x \neq y$ such that $|x - y| = \varepsilon > 0$, then we can pick n sufficiently large so that $1/(2^n - 1) < \varepsilon$. This means that one of the endpoints of the arcs (that partition the circle) must lie between x and y. Hence there is a periodic point that lies between x and y. Since x and y were arbitary choices, no matter which two (distinct) points we pick there will always be a periodic point between them. Hence periodic points are dense in S^1.

2 marks

- We see that the set of periodic points are all rational numbers. If we instead choose some irrational point, then we can prove that it is not-periodic.
  - Pick $\varphi \in S^1 \setminus \mathbb{Q}$. If φ is periodic or eventually periodic then $\exists n, m$ such that $2^n\varphi = 2^m\varphi \mod 1$.

$$\begin{aligned} 2^n\varphi - 2^m\varphi &= k \in \mathbb{Z} \\ \varphi &= \frac{k}{2^n - 2^m} \end{aligned}$$

    which contradicts the irrationality of φ and so it cannot be periodic nor eventually periodic.
  - We can now proceed by noting that the irrational numbers are dense in the reals and so by the above construction, the irrational numbers for a dense set of non-periodic points in S^1.
  - Or if we want a little more detail (not assuming any density results) — Pick $x < y \in S^1$. By writing (and truncating) the decimal expansion of x and y we can find two rational numbers, $\bar{x} < \bar{y}$, between x and y. We can construct an irrational number ϕ between $\bar{x}$ and $\bar{y}$:

$$\phi = \bar{x} + (\bar{y} - \bar{x})/\sqrt{2}$$

    We know (from above) that this point is neither periodic nor eventually periodic. Since our choices of x and y were arbitary, it follows that the set of aperiodic points is dense in S^1.

2 marks

◁ ◁ ◇ ▷ ▷

3. An experimental investigation of rates of convergence: Using a computer investigate (numerically) how quickly an orbit is attracted to a fixed point.

   Procedure: Each of the functions listed below has a fixed point and the orbit of $x_0 = 0.2$ is attracted to it. For each function listed below use a computer (I have provided an applet on the subject homepage) to compute the orbit of $x_0 = 0.2$ until it reaches the fixed point — or within 10^{-5} of it.

For each function you should make note of:

(a) the location of the fixed point, p,

(b) the derivative at the fixed point, $f'(p)$,

(c) is the fixed point attracting or neutral,

(d) the number of iterations it took for the orbit of 0.2 to reach (within 10^{-5}) p.

The functions in question are:

(a) $f(x) = x^2 + 0.25$

(b) $f(x) = x^2$

(c) $f(x) = x^2 - 0.26$

(d) $f(x) = x^2 - 0.75$

(e) $f(x) = 0.4x(1 - x)$

(f) $f(x) = x(1 - x)$

(g) $f(x) = 1.6x(1 - x)$

(h) $f(x) = 2x(1 - x)$

(i) $f(x) = 2.4x(1 - x)$

(j) $f(x) = 3x(1 - x)$

(k) $f(x) = 0.4 \sin x$

(l) $f(x) = \sin x$

Results: When you have collected the data, compare each of the functions. Describe what you observe — in particular the relationship between the speed of convergence and $f'(p)$.

Total = 5 marks

- We first construct a table of all the relevant information:

Function	fixed point	$f'(x)$	iterations
$x^2 + 0.25$	0.5	1	99987
x^2	0	0	3
$x^2 - 0.26$	-0.214142...	-0.428285...	9
$x^2 - 0.75$	-0.5	-1	**lots!**
$0.4x(1-x)$	0	0.4	11
$x(1-x)$	0	1	99985
$1.6x(1-x)$	0.375	0.4	13
$2x(1-x)$	0.5	0	5
$2.4x(1-x)$	0.58333...	-0.4	11
$3x(1-x)$	0.66666...	-1	**lots!** (548240689)
$0.4 \sin x$	0	0.4	11
$\sin x$	0	1	**lots!**

2 marks

- From the table we immediately see that the closer the absolute value of the derivative is to 1 the longer the system takes to converge to the fixed point.
  - If the derivative is 0 then convergence is very fast — less than 10 steps
  - If $|f'(x)| < 1$ then convergence is fast — around 10 steps.
  - If $f'(x) = \pm 1$ then the convergence is slow — but there appears to be a lot of variation. Both $\sin(x)$ and $x^2 - 0.75$ seem to take an eternity to converge — this is because around $x = 0$, $\sin(x)$ is extremely well approximated by $\sin(x) \approx x$. Similarly around $x = -0.5$, $x^2 - 0.75$ is well approximated by $-1 - x$. Whereas the other functions with derivative 1 are less linear.

3 marks